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Planes and Spheres as Topological Manifolds. Stereographic Projection

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Summary. The goal of this article is to show some examples of topological manifolds: planes and spheres in Euclidean space. In doing it, the article introduces the stereographic projection [25].

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The papers [29], [34], [9], [14], [40], [41], [11], [10], [4], [2], [18], [13], [31], [20], [21], [30], [32], [16], [17], [35], [26], [1], [22], [38], [36], [24], [19], [37], [28], [6], [15], [8], [27], [39], [3], [42], [12], [23], [7], [5], and [33] provide the notation and terminology for this paper.

1. Preliminaries

Let us observe that $\emptyset$ is $\emptyset$-valued and $\emptyset$ is onto.

Next we state three propositions:

(1) For every function f and for every set Y holds $\operatorname{dom}(Y \upharpoonright f) = f^{-1}(Y)$.

(2) For every function f and for all sets Y_1, Y_2 such that $Y_2 \subseteq Y_1$ holds $(Y_1 \upharpoonright f)^{-1}(Y_2) = f^{-1}(Y_2)$.

(3) Let S, T be topological structures and f be a function from S into T. If f is homeomorphism, then f^{-1} is homeomorphism.

Let S, T be topological structures. Let us note that the predicate S and T are homeomorphic is symmetric.

For simplicity, we use the following convention: T_1, T_2, T_3 denote topological spaces, A_1 denotes a subset of T_1, A_2 denotes a subset of T_2, and A_3 denotes a subset of T_3.

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Next we state several propositions:

(4) Let f be a function from T_1 into T_2. Suppose f is homeomorphism. Let g be a function from $T_1 \restriction f^{-1}(A_2)$ into $T_2 \restriction A_2$. If $g = A_2 \restriction f$, then g is homeomorphism.

(5) For every function f from T_1 into T_2 such that f is homeomorphism holds $f^{-1}(A_2)$ and A_2 are homeomorphic.

(6) If A_1 and A_2 are homeomorphic, then A_2 and A_1 are homeomorphic.

(7) If A_1 and A_2 are homeomorphic, then A_1 is empty iff A_2 is empty.

(8) If A_1 and A_2 are homeomorphic and A_2 and A_3 are homeomorphic, then A_1 and A_3 are homeomorphic.

(9) If T_1 is second-countable and T_1 and T_2 are homeomorphic, then T_2 is second-countable.

In the sequel n, k are natural numbers and M, N are non empty topological spaces.

The following propositions are true:

(10) If M is Hausdorff and M and N are homeomorphic, then N is Hausdorff.

(11) If M is n-locally Euclidean and M and N are homeomorphic, then N is n-locally Euclidean.

(12) If M is n-manifold and M and N are homeomorphic, then N is n-manifold.

(13) Let x_1, x_2 be finite sequences of elements of $\mathbb{R}$ and i be an element of $\mathbb{N}$. If $i \in \operatorname{dom}(x_1 \bullet x_2)$, then $(x_1 \bullet x_2)(i) = (x_1)_i \cdot (x_2)_i$ and $(x_1 \bullet x_2)_i = (x_1)_i \cdot (x_2)_i$.

(14) For all finite sequences x_1, x_2, y_1, y_2 of elements of $\mathbb{R}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ and $\operatorname{len} y_1 = \operatorname{len} y_2$ holds $x_1 ^\frown y_1 \bullet x_2 ^\frown y_2 = (x_1 \bullet x_2) ^\frown (y_1 \bullet y_2)$.

(15) For all finite sequences x_1, x_2, y_1, y_2 of elements of $\mathbb{R}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ and $\operatorname{len} y_1 = \operatorname{len} y_2$ holds $|(x_1 ^\frown y_1, x_2 ^\frown y_2)| = |(x_1, x_2)| + |(y_1, y_2)|$.

In the sequel p, q, p_1 are points of $\mathcal{E}^n_{\mathrm{T}}$ and r is a real number.

One can prove the following propositions:

(16) If $k \in \operatorname{Seg} n$, then $(p_1 + p_2)(k) = p_1(k) + p_2(k)$.

(17) For every set X holds X is a linear combination of $\mathbb{R}_{\mathbb{R}}^{\operatorname{Seg} n}$ iff X is a linear combination of $\mathcal{E}^n_{\mathrm{T}}$.

(18) Let F be a finite sequence of elements of $\mathcal{E}^n_{\mathrm{T}}$, f_1 be a function from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathbb{R}$, F_1 be a finite sequence of elements of $\mathbb{R}_{\mathbb{R}}^{\operatorname{Seg} n}$, and f_2 be a function from $\mathbb{R}_{\mathbb{R}}^{\operatorname{Seg} n}$ into $\mathbb{R}$. If $f_1 = f_2$ and $F = F_1$, then $f_1 \cdot F = f_2 \cdot F_1$.

(19) Let F be a finite sequence of elements of $\mathcal{E}^n_{\mathrm{T}}$ and F_1 be a finite sequence of elements of $\mathbb{R}_{\mathbb{R}}^{\operatorname{Seg} n}$. If $F_1 = F$, then $\sum F = \sum F_1$.

(20) For every linear combination L_2 of $\mathbb{R}_{\mathbb{R}}^{\operatorname{Seg} n}$ and for every linear combination L_1 of $\mathcal{E}^n_{\mathrm{T}}$ such that $L_1 = L_2$ holds $\sum L_1 = \sum L_2$.

(21) Let A_4 be a subset of $\mathbb{R}_{\mathbb{R}}^{\mathrm{Seg}\,n}$ and A_5 be a subset of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $A_4 = A_5$. Then A_4 is linearly independent if and only if A_5 is linearly independent.

(22) For every subset V of $\mathcal{E}^n_{\mathrm{T}}$ such that $V = \mathbb{R}\text{N-Base}\,n$ there exists a linear combination l of V such that $p = \sum l$.

(23) $\mathbb{R}\text{N-Base}\,n$ is a basis of $\mathcal{E}^n_{\mathrm{T}}$.

(24) Let V be a subset of $\mathcal{E}^n_{\mathrm{T}}$. Then $V \in$ the topology of $\mathcal{E}^n_{\mathrm{T}}$ if and only if for every p such that $p \in V$ there exists r such that $r > 0$ and $\mathrm{Ball}(p, r) \subseteq V$.

Let n be a natural number and let p be a point of $\mathcal{E}^n_{\mathrm{T}}$.

The functor $\mathrm{InnerProduct}\,p$ yields a function from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$ and is defined by:

(Def. 1) For every point q of $\mathcal{E}^n_{\mathrm{T}}$ holds $(\mathrm{InnerProduct}\,p)(q) = |(p, q)|$.

Let us consider n, p. Note that $\mathrm{InnerProduct}\,p$ is continuous.

2. PLANES

Let us consider n and let us consider p, q. The functor $\mathrm{Plane}(p, q)$ yielding a subset of $\mathcal{E}^n_{\mathrm{T}}$ is defined as follows:

(Def. 2) $\mathrm{Plane}(p, q) = \{y; y$ ranges over points of $\mathcal{E}^n_{\mathrm{T}}$: $|(p, y - q)| = 0\}$.

The following propositions are true:

(25) $(\mathrm{transl}(p_1, \mathcal{E}^n_{\mathrm{T}}))^{\circ}\, \mathrm{Plane}(p, p_2) = \mathrm{Plane}(p, p_1 + p_2)$.

(26) If $p \neq 0_{\mathcal{E}^n_{\mathrm{T}}}$, then there exists a linearly independent subset A of $\mathcal{E}^n_{\mathrm{T}}$ such that $\overline{\overline{A}} = n - 1$ and $\Omega_{\mathrm{Lin}(A)} = \mathrm{Plane}(p, 0_{\mathcal{E}^n_{\mathrm{T}}})$.

(27) If $p_1 \neq 0_{\mathcal{E}^n_{\mathrm{T}}}$ and $p_2 \neq 0_{\mathcal{E}^n_{\mathrm{T}}}$, then there exists a function R from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathcal{E}^n_{\mathrm{T}}$ such that R is homeomorphism and $R^{\circ}\, \mathrm{Plane}(p_1, 0_{\mathcal{E}^n_{\mathrm{T}}}) = \mathrm{Plane}(p_2, 0_{\mathcal{E}^n_{\mathrm{T}}})$.

Let us consider n and let us consider p, q. The functor $\mathrm{TPlane}(p, q)$ yields a non empty subspace of $\mathcal{E}^n_{\mathrm{T}}$ and is defined by:

(Def. 3) $\mathrm{TPlane}(p, q) = \mathcal{E}^n_{\mathrm{T}} \restriction \mathrm{Plane}(p, q)$.

The following three propositions are true:

(28) The base finite sequence of $n + 1$ and $n + 1 = (0_{\mathcal{E}^n_{\mathrm{T}}}) \frown \langle 1 \rangle$.

(29) For all points p, q of $\mathcal{E}^{n+1}_{\mathrm{T}}$ such that $p \neq 0_{\mathcal{E}^{n+1}_{\mathrm{T}}}$ holds $\mathcal{E}^n_{\mathrm{T}}$ and $\mathrm{TPlane}(p, q)$ are homeomorphic.

(30) For all points p, q of $\mathcal{E}^{n+1}_{\mathrm{T}}$ such that $p \neq 0_{\mathcal{E}^{n+1}_{\mathrm{T}}}$ holds $\mathrm{TPlane}(p, q)$ is n-manifold.

3. Spheres

Let us consider n. The functor $\mathbb{S}^n$ yields a topological space and is defined by:

(Def. 4) $\mathbb{S}^n = \mathrm{TopUnitCircle}(n+1)$.

Let us consider n. Note that $\mathbb{S}^n$ is non empty.

Let us consider n, p and let S be a subspace of $\mathcal{E}^n_{\mathrm{T}}$. Let us assume that $p \in \mathrm{Sphere}((0_{\mathcal{E}^n_{\mathrm{T}}}), 1)$. The functor $\sigma_{S,p}$ yielding a function from S into $\mathrm{TPlane}(p, 0_{\mathcal{E}^n_{\mathrm{T}}})$ is defined as follows:

(Def. 5) For every q such that $q \in S$ holds $(\sigma_{S,p})(q) = \frac{1}{1-|(q,p)|} \cdot (q - |(q,p)| \cdot p)$.

Next we state the proposition

(31) For every subspace S of $\mathcal{E}^n_{\mathrm{T}}$ such that $\Omega_S = \mathrm{Sphere}((0_{\mathcal{E}^n_{\mathrm{T}}}), 1) \setminus \{p\}$ and $p \in \mathrm{Sphere}((0_{\mathcal{E}^n_{\mathrm{T}}}), 1)$ holds $\sigma_{S,p}$ is homeomorphism.

Let us consider n. One can verify the following observations:

- $\mathbb{S}^n$ is second-countable,
- $\mathbb{S}^n$ is n-locally Euclidean, and
- $\mathbb{S}^n$ is n-manifold.

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