Differentiable Functions on Normed Linear Spaces

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Summary. In this article, we formalize differentiability of functions on normed linear spaces. Partial derivative, mean value theorem for vector-valued functions, continuous differentiability, etc. are formalized. As it is well known, there is no exact analog of the mean value theorem for vector-valued functions. However a certain type of generalization of the mean value theorem for vector-valued functions is obtained as follows: If \(|f'(x + t \cdot h)||\) is bounded for \(t\) between 0 and 1 by some constant \(M\), then \(|f(x + t \cdot h) - f(x)|| \leq M \cdot ||h||\). This theorem is called the mean value theorem for vector-valued functions. By this theorem, the relation between the (total) derivative and the partial derivatives of a function is derived [23].

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The notation and terminology used here have been introduced in the following papers: [28], [29], [9], [4], [30], [12], [10], [25], [11], [1], [2], [26], [7], [3], [5], [8], [17], [22], [20], [27], [21], [31], [14], [24], [18], [16], [15], [19], [13], and [6].

1. Preliminaries

In this paper \(r\) is a real number and \(S, T\) are non trivial real normed spaces. Next we state several propositions:

(1) Let \(R\) be a function from \(\mathbb{R}\) into \(S\). Then \(R\) is rest-like if and only if for every real number \(r\) such that \(r > 0\) there exists a real number \(d\) such that \(d > 0\) and for every real number \(z\) such that \(z \neq 0\) and \(|z| < d\) holds \(|z|^{-1} \cdot ||Rz|| < r\).
(2) Let $R$ be a rest of $S$. Suppose $R_0 = 0_S$. Let $e$ be a real number. Suppose $e > 0$. Then there exists a real number $d$ such that $d > 0$ and for every real number $h$ such that $|h| < d$ holds $\|R_h\| \leq e \cdot |h|$.

(3) For every rest $R$ of $S$ and for every bounded linear operator $L$ from $S$ into $T$ holds $L \cdot R$ is a rest of $T$.

(4) Let $R_1$ be a rest of $S$. Suppose $(R_1)_0 = 0_S$. Let $R_2$ be a rest of $S$, $T$. If $(R_2)_0 = 0_T$, then for every linear $L$ of $S$ holds $R_2 \cdot (L + R_1)$ is a rest of $T$.

(5) Let $R_1$ be a rest of $S$. Suppose $(R_1)_0 = 0_S$. Let $R_2$ be a rest of $S$, $T$. Suppose $(R_2)_0 = 0_T$. Let $L_1$ be a linear of $S$ and $L_2$ be a bounded linear operator from $S$ into $T$. Then $L_2 \cdot R_1 + R_2 \cdot (L_1 + R_1)$ is a rest of $T$.

(6) Let $x_0$ be an element of $\mathbb{R}$ and $g$ be a partial function from $\mathbb{R}$ to the carrier of $S$. Suppose $g$ is differentiable in $x_0$. Let $f$ be a partial function from the carrier of $S$ to the carrier of $T$. Suppose $f$ is differentiable in $g_{x_0}$. Then $f \cdot g$ is differentiable in $x_0$ and $(f \cdot g)'(x_0) = f'(g_{x_0})(g'(x_0))$.

(7) Let $S$ be a real normed space, $x_1$ be a finite sequence of elements of $S$, and $y_1$ be a finite sequence of elements of $\mathbb{R}$. Suppose $\text{len} x_1 = \text{len} y_1$ and for every element $i$ of $\mathbb{N}$ such that $i \in \text{dom} x_1$ holds $y_1(i) = \|(x_1)_i\|$. Then $\|\sum x_1\| \leq \sum y_1$.

(8) Let $S$ be a real normed space, $x$ be a point of $S$, and $N_1, N_2$ be neighbourhoods of $x$. Then $N_1 \cap N_2$ is a neighbourhood of $x$.

(9) For every non-empty finite sequence $X$ and for every set $x$ such that $x \in \Pi X$ holds $x$ is a finite sequence.

Let $G$ be a real norm space sequence. One can verify that $\Pi G$ is constituted finite sequences.

Let $G$ be a real linear space sequence, let $z$ be an element of $\Pi \overline{G}$, and let $j$ be an element of $\text{dom} G$. Then $z(j)$ is an element of $G(j)$.

One can prove the following propositions:

(10) The carrier of $\Pi G = \Pi \overline{G}$.

(11) Let $i$ be an element of $\text{dom} G$, $r$ be a set, and $x$ be a function. If $r \in \text{the carrier of } G(i)$ and $x \in \Pi \overline{G}$, then $x + \cdot (i, r) \in \text{the carrier of } \Pi G$.

Let $G$ be a real norm space sequence. We say that $G$ is nontrivial if and only if:

(Def. 1) For every element $j$ of $\text{dom } G$ holds $G(j)$ is non trivial.

Let us mention that there exists a real norm space sequence which is nontrivial.

Let $G$ be a nontrivial real norm space sequence and let $i$ be an element of $\text{dom } G$. Note that $G(i)$ is non trivial.

Let $G$ be a nontrivial real norm space sequence. Note that $\Pi G$ is non trivial.

The following propositions are true:
(12) Let $G$ be a real norm space sequence, $p$, $q$ be points of $\prod G$, and $r_0$, $p_0$, $q_0$ be elements of $\prod G$. Suppose $p = p_0$ and $q = q_0$. Then $p + q = r_0$ if and only if for every element $i$ of dom $G$ holds $r_0(i) = p_0(i) + q_0(i)$.

(13) Let $G$ be a real norm space sequence, $p$ be a point of $\prod G$, $r$ be a real number, and $r_0$, $p_0$ be elements of $\prod G$. Suppose $p = p_0$. Then $r \cdot p = r_0$ if and only if for every element $i$ of dom $G$ holds $r_0(i) = r \cdot p_0(i)$.

(14) Let $G$ be a real norm space sequence and $p_0$ be an element of $\prod G$. Then $0\prod G = p_0$ if and only if for every element $i$ of dom $G$ holds $p_0(i) = 0_{G(i)}$.

(15) Let $G$ be a real norm space sequence, $p$, $q$ be points of $\prod G$, and $r_0$, $p_0$, $q_0$ be elements of $\prod G$. Suppose $p = p_0$ and $q = q_0$. Then $p - q = r_0$ if and only if for every element $i$ of dom $G$ holds $r_0(i) = p_0(i) - q_0(i)$.

2. Mean Value Theorem for Vector-Valued Functions

Let $S$ be a real linear space and let $p$, $q$ be points of $S$. The functor $]p, q[$ yielding a subset of $S$ is defined as follows:

(Def. 2) $]p, q[ = \{p + t \cdot (q - p); t \text{ ranges over real numbers: } 0 < t \land t < 1\}$.

Let $S$ be a real linear space and let $p$, $q$ be points of $S$. We introduce $[p, q]$ as a synonym of $\mathcal{L}(p, q)$.

Next we state several propositions:

(16) For every real linear space $S$ and for all points $p$, $q$ of $S$ holds $]p, q[ \subseteq [p, q]$.

(17) Let $T$ be a non trivial real normed space and $R$ be a partial function from $\mathbb{R}$ to $T$. Suppose $R$ is total. Then $R$ is rest-like if and only if for every real number $r$ such that $r > 0$ there exists a real number $d$ such that $d > 0$ and for every real number $z$ such that $z \neq 0$ and $|z| < d$ holds $\frac{\|R(z)\|}{|z|} < r$.

(18) Let $R$ be a function from $\mathbb{R}$ into $\mathbb{R}$. Then $R$ is rest-like if and only if for every real number $r$ such that $r > 0$ there exists a real number $d$ such that $d > 0$ and for every real number $z$ such that $z \neq 0$ and $|z| < d$ holds $\frac{|R(z)|}{|z|} < r$.

(19) Let $S$, $T$ be non trivial real normed spaces, $f$ be a partial function from $S$ to $T$, $p$, $q$ be points of $S$, and $M$ be a real number. Suppose that

(i) $]p, q[ \subseteq \text{dom } f$,
(ii) for every point $x$ of $S$ such that $x \in ]p, q]$ holds $f$ is continuous in $x$,
(iii) for every point $x$ of $S$ such that $x \in ]p, q]$ holds $f$ is differentiable in $x$, and
(iv) for every point $x$ of $S$ such that $x \in ]p, q]$ holds $\|f'(x)\| \leq M$.

Then $\|f_q - f_p\| \leq M \cdot \|q - p\|$.
(20) Let $S, T$ be non trivial real normed spaces, $f$ be a partial function from $S$ to $T$, $p, q$ be points of $S$, $M$ be a real number, and $L$ be a point of the real norm space of bounded linear operators from $S$ into $T$. Suppose that

(i) $[p, q] \subseteq \text{dom } f$,

(ii) for every point $x$ of $S$ such that $x \in [p, q]$ holds $f$ is continuous in $x$,

(iii) for every point $x$ of $S$ such that $x \in ]p, q[ \text{ holds } f$ is differentiable in $x$, and

(iv) for every point $x$ of $S$ such that $x \in ]p, q[$ holds $\|f'(x) - L\| \leq M$.

Then $\|f_q - f_p - L(q - p)\| \leq M \cdot \|q - p\|$.

3. Partial Derivative of a Function of Several Variables

Let $G$ be a real norm space sequence and let $i$ be an element of $\text{dom } G$. The projection onto $i$ yielding a function from $\prod G$ into $G(i)$ is defined by:

(Def. 3) For every element $x$ of $\prod G$ holds $(\text{the projection onto } i)(x) = x(i)$.

Let $G$ be a real norm space sequence, let $i$ be an element of $\text{dom } G$, and let $x$ be an element of $\prod G$. The functor reproj($i, x$) yielding a function from $G(i)$ into $\prod G$ is defined by:

(Def. 4) For every element $r$ of $G(i)$ holds $(\text{reproj}(i, x))(r) = x + \cdot (i, r)$.

Let $G$ be a nontrivial real norm space sequence and let $j$ be a set. Let us assume that $j \in \text{dom } G$. The functor modetrans($G, j$) yields an element of $\text{dom } G$ and is defined by:

(Def. 5) $\text{modetrans}(G, j) = j$.

Let $G$ be a nontrivial real norm space sequence, let $F$ be a non trivial real normed space, let $i$ be a set, let $f$ be a partial function from $\prod G$ to $F$, and let $x$ be an element of $\prod G$. We say that $f$ is partially differentiable in $x$ w.r.t. $i$ if and only if:

(Def. 6) $f \cdot \text{reproj(}\text{modetrans}(G, i), x)\text{ is differentiable in (the projection onto } \text{modetrans}(G, i))(x)$.

Let $G$ be a nontrivial real norm space sequence, let $F$ be a non trivial real normed space, let $i$ be a set, let $f$ be a partial function from $\prod G$ to $F$, and let $x$ be a point of $\prod G$. The functor partdiff($f, x, i$) yielding a point of the real norm space of bounded linear operators from $G(\text{modetrans}(G, i))$ into $F$ is defined as follows:

(Def. 7) $\text{partdiff}(f, x, i) = f \cdot \text{reproj(}\text{modetrans}(G, i), x)'((\text{the projection onto } \text{modetrans}(G, i))(x))$. 
4. Linearity of Partial Differential Operator

For simplicity, we adopt the following rules: $G$ denotes a nontrivial real norm space sequence, $F$ denotes a non trivial real normed space, $i$ denotes an element of $\text{dom} \, G$, $f$, $f_1$, $f_2$ denote partial functions from $\prod G$ to $F$, $x$ denotes a point of $\prod G$, and $X$ denotes a set.

Let $G$ be a nontrivial real norm space sequence, let $F$ be a non trivial real normed space, let $i$ be a set, let $f$ be a partial function from $\prod G$ to $F$, and let $X$ be a set. We say that $f$ is partially differentiable on $X \, w.r.t. \, i$ if and only if:

(Def. 8) $X \subseteq \text{dom} \, f$ and for every point $x$ of $\prod G$ such that $x \in X$ holds $f \mid X$ is partially differentiable in $x \, w.r.t. \, i$.

Next we state several propositions:

(21) For every element $x_2$ of $G(i)$ holds $\|(\text{reproj}(i,0)G)(x_2)\| = \|x_2\|$. 

(22) Let $G$ be a nontrivial real norm space sequence, $i$ be an element of $\text{dom} \, G$, $x$ be a point of $\prod G$, and $r$ be a point of $G(i)$. Then $(\text{reproj}(i,x))(r)−x = (\text{reproj}(i,0)G)(r−(\text{the projection onto } i/x))$ and $x−(\text{reproj}(i,x))(r) = (\text{reproj}(i,0)G)((\text{the projection onto } i/x)−r)$.

(23) Let $G$ be a nontrivial real norm space sequence, $i$ be an element of $\text{dom} \, G$, $x$ be a point of $\prod G$, and $Z$ be a subset of $\prod G$. Suppose $Z$ is open and $x \in Z$. Then there exists a neighbourhood $N$ of $(\text{the projection onto } i/x)$ such that for every point $z$ of $G(i)$ if $z \in N$, then $(\text{reproj}(i,x))(z) \in Z$.

(24) Let $G$ be a nontrivial real norm space sequence, $T$ be a non trivial real normed space, $i$ be a set, $f$ be a partial function from $\prod G$ to $T$, and $Z$ be a subset of $\prod G$. Suppose $Z$ is open. Then $f$ is partially differentiable on $Z \, w.r.t. \, i$ if and only if $Z \subseteq \text{dom} \, f$ and for every point $x$ of $\prod G$ such that $x \in Z$ holds $f$ is partially differentiable in $x \, w.r.t. \, i$.

(25) For every set $i$ such that $i \in \text{dom} \, G$ and $f$ is partially differentiable on $X \, w.r.t. \, i$ holds $X$ is a subset of $\prod G$.

Let $G$ be a nontrivial real norm space sequence, let $S$ be a non trivial real normed space, and let $i$ be a set. Let us assume that $i \in \text{dom} \, G$. Let $f$ be a partial function from $\prod G$ to $S$ and let $X$ be a set. Let us assume that $f$ is partially differentiable on $X \, w.r.t. \, i$. The functor $f \mid X$ yields a partial function from $\prod G$ to the real norm space of bounded linear operators from $G(\text{modetrans}(G,i))$ into $S$ and is defined by:

(Def. 9) $\text{dom}(f \mid X) = X$ and for every point $x$ of $\prod G$ such that $x \in X$ holds $(f \mid X)_x = \text{partdiff}(f, x, i)$.

One can prove the following propositions:

(26) For every set $i$ such that $i \in \text{dom} \, G$ holds $(f_1 + f_2) \cdot \text{reproj}(\text{modetrans}(G,i), x) = f_1 \cdot \text{reproj}(\text{modetrans}(G,i), x) + f_2 \cdot \text{reproj}(\text{modetrans}(G,i), x)$.
reproj(modetrans(G, i, x) and \((f_1 - f_2) \cdot \text{reproj(modetrans}(G, i, x) = f_1 \cdot \text{reproj(modetrans}(G, i, x) - f_2 \cdot \text{reproj(modetrans}(G, i, x).

(27) For every set \(i\) such that \(i \in \text{dom } G\) holds \(r \cdot f \cdot \text{reproj(modetrans}(G, i, x) = (r \cdot f) \cdot \text{reproj(modetrans}(G, i, x).

(28) Let \(i\) be a set. Suppose \(i \in \text{dom } G\) and \(f_1\) is partially differentiable in \(x\) w.r.t. \(i\) and \(f_2\) is partially differentiable in \(x\) w.r.t. \(i\). Then \(f_1 + f_2\) is partially differentiable in \(x\) w.r.t. \(i\) and \(\text{partdiff}(f_1 + f_2, x, i) = \text{partdiff}(f_1, x, i) + \text{partdiff}(f_2, x, i).

(29) Let \(i\) be a set. Suppose \(i \in \text{dom } G\) and \(f_1\) is partially differentiable in \(x\) w.r.t. \(i\) and \(f_2\) is partially differentiable in \(x\) w.r.t. \(i\). Then \(f_1 - f_2\) is partially differentiable in \(x\) w.r.t. \(i\) and \(\text{partdiff}(f_1 - f_2, x, i) = \text{partdiff}(f_1, x, i) - \text{partdiff}(f_2, x, i).

(30) Let \(i\) be a set. Suppose \(i \in \text{dom } G\) and \(f\) is partially differentiable in \(x\) w.r.t. \(i\). Then \(r \cdot f\) is partially differentiable in \(x\) w.r.t. \(i\) and \(\text{partdiff}(r \cdot f, x, i) = r \cdot \text{partdiff}(f, x, i).

5. Continuous Differentiability of Partial Derivative

Next we state the proposition

(31) \(\|(\text{the projection onto } i)(x)\| \leq \|x\|\).

Let \(G\) be a nontrivial real norm space sequence. One can verify that every point of \(\prod G\) is len-\(G\)-element.

We now state a number of propositions:

(32) Let \(G\) be a nontrivial real norm space sequence, \(T\) be a non trivial real normed space, \(i\) be a set, \(Z\) be a subset of \(\prod G\), and \(f\) be a partial function from \(\prod G\) to \(T\). Suppose \(Z\) is open. Then \(f\) is partially differentiable on \(Z\) w.r.t. \(i\) if and only if \(Z \subseteq \text{dom } f\) and for every point \(x\) of \(\prod G\) such that \(x \in Z\) holds \(f\) is partially differentiable in \(x\) w.r.t. \(i\).

(33) Let \(i, j\) be elements of dom \(G\), \(x\) be a point of \(G(i)\), and \(z\) be an element of \(\prod G\) such that \(z = (\text{reproj}(i, 0\prod G))(x)\). Then
   (i) if \(i = j\), then \(z(j) = x\), and
   (ii) if \(i \neq j\), then \(z(j) = 0_{G(j)}\).

(34) For all points \(x, y\) of \(G(i)\) holds \((\text{reproj}(i, 0\prod G))(x + y) = (\text{reproj}(i, 0\prod G))(x) + (\text{reproj}(i, 0\prod G))(y)\).

(35) Let \(x, y\) be points of \(\prod G\). Then \((\text{the projection onto } i)(x + y) = (\text{the projection onto } i)(x) + (\text{the projection onto } i)(y)\).

(36) For all points \(x, y\) of \(G(i)\) holds \((\text{reproj}(i, 0\prod G))(x - y) = (\text{reproj}(i, 0\prod G))(x) - (\text{reproj}(i, 0\prod G))(y)\).
Let $x, y$ be points of $\prod G$. Then (the projection onto $i$)(x−y) = (the projection onto $i$)(x)−(the projection onto $i$)(y).

For every point $x$ of $G(i)$ such that $x \neq 0_{G(i)}$ holds (reproj$(i, 0_{\prod G})$)(x) ≠ 0$_{\prod G}$.

For every point $x$ of $G(i)$ and for every element $a$ of $\mathbb{R}$ holds (reproj$(i, 0_{\prod G})$)(a·x) = a·(reproj$(i, 0_{\prod G})$)(x).

Let $x$ be a point of $\prod G$ and $a$ be an element of $\mathbb{R}$. Then (the projection onto $i$)(a·x) = a·(the projection onto $i$)(x).

Let $G$ be a nontrivial real norm space sequence, $S$ be a non trivial real normed space, $f$ be a partial function from $\prod G$ to $S$, $x$ be a point of $\prod G$, and $i$ be a set. Suppose $f$ is differentiable in $x$. Then $f$ is partially differentiable in $x$ w.r.t. $i$ and partdiff$(f, x, i) = f'(x) \cdot$ reproj$[\text{modetrans}(G, i), 0_{\prod G})$.

Let $S$ be a real normed space and $h, g$ be finite sequences of elements of $S$. Suppose $\text{len } h = \text{len } g + 1$ and for every natural number $i$ such that $i \in \text{dom } g$ holds $g_i = h_i - h_{i+1}$. Then $h_1 - h_{\text{len } h} = \sum g$.

Let $G$ be a nontrivial real norm space sequence, $x, y$ be elements of $\prod \mathcal{G}$, and $Z$ be a set. Then $x + y | Z$ is an element of $\prod \mathcal{G}$.

Let $G$ be a nontrivial real norm space sequence, $x, y$ be points of $\prod G$, $Z, x_0$ be elements of $\prod \mathcal{G}$, and $X$ be a set. If $Z = 0_{\prod G}$ and $x_0 = x$ and $y = Z + x_0 | X$, then $||y|| \leq ||x||$.

Let $G$ be a nontrivial real norm space sequence, $S$ be a non trivial real normed space, $f$ be a partial function from $\prod G$ to $S$, and $x, y$ be points of $\prod G$. Then there exists a finite sequence $h$ of elements of $\prod G$ and there exists a finite sequence $g$ of elements of $S$ and there exist elements $Z, y_0$ of $\prod \mathcal{G}$ such that $y_0 = y$ and $Z = 0_{\prod G}$ and $\text{len } h = \text{len } G + 1$ and $\text{len } g = \text{len } G$ and for every natural number $i$ such that $i \in \text{dom } h$ holds $h_i = Z + y_0 | \text{Seg}(\text{len } G + 1)^i$ and for every natural number $i$ such that $i \in \text{dom } g$ holds $g_i = f_{x+h_i} - f_{x+h_{i+1}}$ and for every natural number $i$ and for every point $h_1$ of $\prod G$ such that $i \in \text{dom } h$ and $h_i = h_1$ holds $\|h_1\| \leq ||y||$ and $f_{x+y} - f_x = \sum g$.

Let $G$ be a nontrivial real norm space sequence, $i$ be an element of dom $G$, $x, y$ be points of $\prod G$, and $x_2$ be a point of $G(i)$. If $y = (\text{reproj}(i, x))(x_2)$, then (the projection onto $i$)(y) = $x_2$.

Let $G$ be a nontrivial real norm space sequence, $i$ be an element of dom $G$, $y$ be a point of $\prod G$, and $q$ be a point of $G(i)$. If $q = (\text{the projection onto } i)(y)$, then $y = (\text{reproj}(i, y))(q)$.

Let $G$ be a nontrivial real norm space sequence, $i$ be an element of dom $G$, $x, y$ be points of $\prod G$, and $x_2$ be a point of $G(i)$. If $y = (\text{reproj}(i, x))(x_2)$, then reproj$(i, x) = \text{reproj}(i, y)$.
(49) Let $G$ be a nontrivial real norm space sequence, $i, j$ be elements of $\text{dom} G$, $x, y$ be points of $\prod G$, and $x_2$ be a point of $G(i)$. Suppose $y = (\text{reproj}(i, x))(x_2)$ and $i \neq j$. Then (the projection onto $j)(x) = (the$ $projection$ $onto$ $j)(y)$.

(50) Let $G$ be a nontrivial real norm space sequence, $F$ be a non trivial real norm space, $i$ be an element of $\text{dom} G$, $x$ be a point of $\prod G$, $x_2$ be a point of $G(i)$, $f$ be a partial function from $\prod G$ to $F$, and $g$ be a partial function from $G(i)$ to $F$. If (the projection onto $i)(x) = x_2$ and $g = f \cdot \text{reproj}(i, x)$, then $g(x_2) = \text{partdiff}(f, x, i)$.

(51) Let $G$ be a nontrivial real norm space sequence, $F$ be a non trivial real norm space sequence, $f$ be a partial function from $\prod G$ to $F$, $x$ be a point of $\prod G$, $i$ be a set, $M$ be a real number, $L$ be a point of the real norm space of bounded linear operators from $G(\text{modetrans}(G, i))$ into $F$, and $p, q$ be points of $G(\text{modetrans}(G, i))$. Suppose that

(i) $i \in \text{dom} G$,

(ii) for every point $h$ of $G(\text{modetrans}(G, i))$ such that $h \in [p, q]$ holds $\|\text{partdiff}(f, \text{reproj}(\text{modetrans}(G, i), x)(h), i) - L\| \leq M$,

(iii) for every point $h$ of $G(\text{modetrans}(G, i))$ such that $h \in [p, q]$ holds $\text{reproj}(\text{modetrans}(G, i), x)(h) \in \text{dom} f$, and

(iv) for every point $h$ of $G(\text{modetrans}(G, i))$ such that $h \in [p, q]$ holds $f$ is partially differentiable in $\text{reproj}(\text{modetrans}(G, i), x)(h)$ w.r.t. $i$.

Then $\|f(\text{reproj}(\text{modetrans}(G, i), x)(q)) - f(\text{reproj}(\text{modetrans}(G, i), x)(p)) - L(q - p)\| \leq M \cdot \|q - p\|$.

(52) Let $G$ be a nontrivial real norm space sequence, $x, y, z, w$ be points of $\prod G$, $i$ be an element of $\text{dom} G$, $d$ be a real number, and $p, q, r$ be points of $G(i)$. Suppose $\|y - x\| < d$ and $\|z - x\| < d$ and $p = (the$ $projection$ $onto$ $i)(y)$ and $z = (\text{reproj}(i, y))(q)$ and $r \in [p, q]$ and $w = (\text{reproj}(i, y))(r)$. Then $\|w - x\| < d$.

(53) Let $G$ be a nontrivial real norm space sequence, $S$ be a non trivial real norm space, $f$ be a partial function from $\prod G$ to $S$, $X$ be a subset of $\prod G$, $x, y, z$ be points of $\prod G$, $i$ be a set, $p, q$ be points of $G(\text{modetrans}(G, i))$, and $d, r$ be real numbers. Suppose that $i \in \text{dom} G$ and $X$ is open and $x \in X$ and $\|y - x\| < d$ and $\|z - x\| < d$ and $X \subseteq \text{dom} f$ and for every point $x$ of $\prod G$ such that $x \in X$ holds $f$ is partially differentiable in $x$ w.r.t. $i$ and for every point $z$ of $\prod G$ such that $\|z - x\| < d$ holds $\|\text{partdiff}(f, z, i) - \text{partdiff}(f, x, i)\| \leq r$ and $z = (\text{reproj}(\text{modetrans}(G, i), y))(p)$ and $q = (the$ $projection$ $onto$ $\text{modetrans}(G, i))(y)$. Then $\|f_z - f_y - (\text{partdiff}(f, x, i))(p - q)\| \leq \|p - q\| \cdot r$.

(54) Let $G$ be a nontrivial real norm space sequence, $h$ be a finite sequence of elements of $\prod G$, $y, x$ be points of $\prod G$, $y_0, Z$ be elements of $\prod G$, and $j$ be an element of $\mathbb{N}$. Suppose $y = y_0$ and $Z = 0\prod G$ and
len \( h \) = \( len G + 1 \) and \( 1 \leq j \leq len G \) and for every natural number \( i \) such that \( i \in dom h \) holds \( h_i = Z+y_0 | Seg((len G + 1) -' i) \). Then \( x + h_j = (reproj(modetrans(G, (len G + 1) -' j), x + h_{j+1}))(\text{(the projection onto modetrans}(G, (len G + 1) -' j))(x + y)). \)

(55) Let \( G \) be a nontrivial real norm space sequence, \( h \) be a finite sequence of elements of \( \prod G, y \) be points of \( \prod G, y_0, Z \) be elements of \( \prod G, and j \) be an element of \( \mathbb{N} \). Suppose \( y = y_0 \) and \( Z = 0\prod G \) and \( len h = len G + 1 \) and \( 1 \leq j \leq len G \) and for every natural number \( i \) such that \( i \in dom h \) holds \( h_i = Z+y_0 | Seg((len G + 1) -' i) \). Then (the projection onto modetrans}(G, (len G + 1) -' j))(x + y) - (the projection onto modetrans}(G, (len G + 1) -' j))(x + h_{j+1}) = (the projection onto modetrans}(G, (len G + 1) -' j))(y).

(56) Let \( G \) be a nontrivial real norm space sequence, \( S \) be a non trivial real normed space, \( f \) be a partial function from \( \prod G \) to \( S \), \( X \) be a subset of \( \prod G \), and \( x \) be a point of \( \prod G \). Suppose that

(i) \( X \) is open,

(ii) \( x \in X \), and

(iii) for every set \( i \) such that \( i \in dom G \) holds \( f \) is partially differentiable on \( X \) w.r.t. \( i \) and \( f^i X \) is continuous on \( X \).

Then

(iv) \( f \) is differentiable in \( x \), and

(v) for every point \( h \) of \( \prod G \) there exists a finite sequence \( w \) of elements of \( S \) such that \( dom w = dom G \) and for every set \( i \) such that \( i \in dom G \) holds \( h(i) = \text{partdiff}(f, x, i))(\text{(the projection onto modetrans}(G, i))(h)) \) and \( f^i(x)(h) = \sum w \).

(57) Let \( G \) be a nontrivial real norm space sequence, \( F \) be a non trivial real normed space, \( f \) be a partial function from \( \prod G \) to \( F \), and \( X \) be a subset of \( \prod G \). Suppose \( X \) is open. Then for every set \( i \) such that \( i \in dom G \) holds \( f \) is partially differentiable on \( X \) w.r.t. \( i \) and \( f^i X \) is continuous on \( X \) if and only if \( f \) is differentiable on \( X \) and \( f^i X \) is continuous on \( X \).

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