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Valuation Theory. Part I

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Summary. In the article we introduce a valuation function over a field [1]. Ring of non negative elements and its ideal of positive elements have been also defined.

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The notation and terminology used here have been introduced in the following papers: [11], [19], [4], [15], [20], [8], [21], [10], [9], [16], [3], [5], [7], [18], [17], [13], [14], [6], [2], and [12].

1. Extended Reals

We use the following convention: x, y, z, s are extended real numbers, i is an integer, and n, m are natural numbers.

The following propositions are true:

(1) If $x = -x$, then $x = 0$.

(2) If $x + x = 0$, then $x = 0$.

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(3) If $0 \leq x \leq y$ and $0 \leq s \leq z$, then $x \cdot s \leq y \cdot z$.

(4) If $y \neq +\infty$ and $0 < x$ and $0 < y$, then $0 < \frac{x}{y}$.

(5) If $y \neq +\infty$ and $x < 0 < y$, then $\frac{x}{y} < 0$.

(6) If $y \neq -\infty$ and $0 < x$ and $y < 0$, then $\frac{x}{y} < 0$.

(7) If $x, y \in \mathbb{R}$ or $z \in \mathbb{R}$, then $\frac{x+y}{z} = \frac{x}{z} + \frac{y}{z}$.

(8) If $y \neq +\infty$ and $y \neq -\infty$ and $y \neq 0$, then $\frac{x}{y} \cdot y = x$.

(9) If $y \neq -\infty$ and $y \neq +\infty$ and $x \neq 0$ and $y \neq 0$, then $\frac{x}{y} \neq 0$.

Let x be a number. We say that x is extended integer if and only if:

(Def. 1) x is integer or $x = +\infty$.

Let us mention that every number which is extended integer is also extended real.

One can verify the following observations:

* $+\infty$ is extended integer,
* $-\infty$ is non extended integer,
* $\overline{1}$ is extended integer, positive, and real,
* every number which is integer is also extended integer, and
* every number which is real and extended integer is also integer.

Let us observe that there exists an element of $\overline{\mathbb{R}}$ which is real, extended integer, and positive and there exists an extended integer number which is positive.

An extended integer is an extended integer number.

In the sequel x, y, v denote extended integers.

One can prove the following propositions:

(10) If $x < y$, then $x + 1 \leq y$.

(11) $-\infty < x$.

Let X be an extended real-membered set. Let us assume that there exists a positive extended integer i_0 such that $i_0 \in X$. The functor least-positive X yielding a positive extended integer is defined by:

(Def. 2) least-positive $X \in X$ and for every positive extended integer i such that $i \in X$ holds least-positive $X \leq i$.

Let f be a binary relation. We say that f is extended integer valued if and only if:

(Def. 3) For every set x such that $x \in \operatorname{rng} f$ holds x is extended integer.

Let us note that there exists a function which is extended integer valued.

Let A be a set. Note that there exists a function from A into $\overline{\mathbb{R}}$ which is extended integer valued.

Let f be an extended integer valued function and let x be a set. Note that $f(x)$ is extended integer.

2. STRUCTURES

One can prove the following proposition

(12) Let K be a distributive left unital add-associative right zeroed right complementable non empty double loop structure. Then $-1_K \cdot -1_K = 1_K$.

Let K be a non empty double loop structure, let S be a subset of K, and let n be a natural number. The functor S^n yielding a subset of K is defined by:

(Def. 4)(i) $S^n =$ the carrier of K if $n = 0$,

(ii) there exists a finite sequence f of elements of $2^{\text{the carrier of } K}$ such that $S^n = f(\operatorname{len} f)$ and $\operatorname{len} f = n$ and $f(1) = S$ and for every natural number i such that i, $i + 1 \in \operatorname{dom} f$ holds $f(i + 1) = S * f_i$, otherwise.

In the sequel A denotes a subset of D. The following propositions are true:

(13) $A^1 = A$.

(14) $A^2 = A * A$.

Let R be a ring, let S be an ideal of R, and let n be a natural number. Observe that S^n is non empty, add closed, left ideal, and right ideal.

Let G be a non empty double loop structure, let g be an element of G, and let i be an integer. The functor g^i yielding an element of G is defined as follows:

(Def. 5) $g^i = \begin{cases} \operatorname{power}_G(g, |i|), & \text{if } 0 \leq i, \\ \operatorname{power}_G(g, |i|)^{-1}, & \text{otherwise.} \end{cases}$

Let G be a non empty double loop structure, let g be an element of G, and let n be a natural number. Then g^n can be characterized by the condition:

(Def. 6) $g^n = \operatorname{power}_G(g, n)$.

In the sequel K is a field-like non degenerated associative add-associative right zeroed right complementable distributive Abelian non empty double loop structure and a, b, c are elements of K. We now state two propositions:

(15) $a^{n+m} = a^n \cdot a^m$.

(16) If $a \neq 0_K$, then $a^i \neq 0_K$.

3. VALUATION

Let K be a double loop structure. We say that K has a valuation if and only if the condition (Def. 7) is satisfied.

(Def. 7) There exists an extended integer valued function f from K into $\overline{\mathbb{R}}$ such that

(i) $f(0_K) = +\infty$,

(ii) for every element a of K such that $a \neq 0_K$ holds $f(a) \in \mathbb{Z}$,

(iii) for all elements a, b of K holds $f(a \cdot b) = f(a) + f(b)$,

(iv) for every element a of K such that $0 \leq f(a)$ holds $0 \leq f(1_K + a)$, and

(v) there exists an element a of K such that $f(a) \neq 0$ and $f(a) \neq +\infty$.

Let K be a double loop structure. Let us assume that K has a valuation. An extended integer valued function from K into $\overline{\mathbb{R}}$ is said to be a valuation of K if it satisfies the conditions (Def. 8).

(Def. 8)(i) $\mathrm{It}(0_K) = +\infty$,

(ii) for every element a of K such that $a \neq 0_K$ holds $\mathrm{it}(a) \in \mathbb{Z}$,

(iii) for all elements a, b of K holds $\mathrm{it}(a \cdot b) = \mathrm{it}(a) + \mathrm{it}(b)$,

(iv) for every element a of K such that $0 \leq \mathrm{it}(a)$ holds $0 \leq \mathrm{it}(1_K + a)$, and

(v) there exists an element a of K such that $\mathrm{it}(a) \neq 0$ and $\mathrm{it}(a) \neq +\infty$.

In the sequel v denotes a valuation of K.

One can prove the following propositions:

(17) If K has a valuation, then $v(1_K) = 0$.

(18) If K has a valuation and $a \neq 0_K$, then $v(a) \neq +\infty$.

(19) If K has a valuation, then $v(-1_K) = 0$.

(20) If K has a valuation, then $v(-a) = v(a)$.

(21) If K has a valuation and $a \neq 0_K$, then $v(a^{-1}) = -v(a)$.

(22) If K has a valuation and $b \neq 0_K$, then $v(\frac{a}{b}) = v(a) - v(b)$.

(23) If K has a valuation and $a \neq 0_K$ and $b \neq 0_K$, then $v(\frac{a}{b}) = -v(\frac{b}{a})$.

(24) If K has a valuation and $b \neq 0_K$ and $0 \leq v(\frac{a}{b})$, then $v(b) \leq v(a)$.

(25) If K has a valuation and $a \neq 0_K$ and $b \neq 0_K$ and $v(\frac{a}{b}) \leq 0$, then $0 \leq v(\frac{b}{a})$.

(26) If K has a valuation and $b \neq 0_K$ and $v(\frac{a}{b}) \leq 0$, then $v(a) \leq v(b)$.

(27) If K has a valuation, then $\min(v(a), v(b)) \leq v(a + b)$.

(28) If K has a valuation and $v(a) < v(b)$, then $v(a) = v(a + b)$.

(29) If K has a valuation and $a \neq 0_K$, then $v(a^i) = i \cdot v(a)$.

(30) If K has a valuation and $0 \leq v(1_K + a)$, then $0 \leq v(a)$.

(31) If K has a valuation and $0 \leq v(1_K - a)$, then $0 \leq v(a)$.

(32) If K has a valuation and $a \neq 0_K$ and $v(a) \leq v(b)$, then $0 \leq v(\frac{b}{a})$.

(33) If K has a valuation, then $+\infty \in \mathrm{rng}\, v$.

(34) If $v(a) = 1$, then $\text{least-positive}\, \mathrm{rng}\, v = 1$.

(35) If K has a valuation, then $\text{least-positive}\, \mathrm{rng}\, v$ is integer.

(36) If K has a valuation, then for every element x of K such that $x \neq 0_K$ there exists an integer i such that $v(x) = i \cdot \text{least-positive}\, \mathrm{rng}\, v$.

Let us consider K, v. Let us assume that K has a valuation. The functor $\mathrm{Pgenerator}\, v$ yielding an element of K is defined as follows:

(Def. 9) $\mathrm{Pgenerator}\, v =$ the element of $v^{-1}(\{\text{least-positive}\, \mathrm{rng}\, v\})$.

Let us consider K, v. Let us assume that K has a valuation. The functor $\text{normal-valuation}\, v$ yields a valuation of K and is defined by:

(Def. 10) $v(a) = (\text{normal-valuation}\, v)(a) \cdot \text{least-positive}\, \mathrm{rng}\, v$.

We now state a number of propositions:

(37) If K has a valuation, then $v(a) = 0$ iff (normal-valuation v)$(a) = 0$.

(38) If K has a valuation, then $v(a) = +\infty$ iff (normal-valuation v)$(a) = +\infty$.

(39) If K has a valuation, then $v(a) = v(b)$ iff (normal-valuation v)(a) = (normal-valuation v)(b).

(40) If K has a valuation, then $v(a)$ is positive iff (normal-valuation v)(a) is positive.

(41) If K has a valuation, then $0 \leq v(a)$ iff $0 \leq$ (normal-valuation v)(a).

(42) If K has a valuation, then $v(a)$ is non negative iff (normal-valuation v)(a) is non negative.

(43) If K has a valuation, then (normal-valuation v)(Pgenerator v) $= 1$.

(44) If K has a valuation and $0 \leq v(a)$, then (normal-valuation v)$(a) \leq v(a)$.

(45) If K has a valuation and $v(a) = 1$, then normal-valuation $v = v$.

(46) If K has a valuation, then normal-valuation(normal-valuation v) = normal-valuation v.

4. Valuation Ring

Let K be a non empty double loop structure and let v be a valuation of K. The functor NonNegElements v is defined as follows:

(Def. 11) NonNegElements $v = \{x \in K : 0 \leq v(x)\}$.

The following four propositions are true:

(47) Let K be a non empty double loop structure, v be a valuation of K, and a be an element of K. Then $a \in$ NonNegElements v if and only if $0 \leq v(a)$.

(48) For every non empty double loop structure K and for every valuation v of K holds NonNegElements $v \subseteq$ the carrier of K.

(49) For every non empty double loop structure K and for every valuation v of K such that K has a valuation holds $0_K \in$ NonNegElements v.

(50) If K has a valuation, then $1_K \in$ NonNegElements v.

Let us consider K, v. Let us assume that K has a valuation. The functor ValuatRing v yields a strict commutative non degenerated ring and is defined by the conditions (Def. 12).

(Def. 12)(i) The carrier of ValuatRing v = NonNegElements v,

(ii) the addition of ValuatRing v = (the addition of K)$\restriction$(NonNegElements $v \times$ NonNegElements v),

(iii) the multiplication of ValuatRing v = (the multiplication of K)$\restriction$(NonNegElements $v \times$ NonNegElements v),

(iv) the zero of ValuatRing $v = 0_K$, and

(v) the one of ValuatRing $v = 1_K$.

The following propositions are true:

(51) If K has a valuation, then every element of ValuatRing v is an element of K.

(52) If K has a valuation, then $0 \leq v(a)$ iff a is an element of ValuatRing v.

(53) If K has a valuation, then for every subset S of ValuatRing v holds 0 is a lower bound of $v^\circ S$.

(54) Suppose K has a valuation. Let x, y be elements of K and x_1, y_1 be elements of ValuatRing v. If $x = x_1$ and $y = y_1$, then $x + y = x_1 + y_1$.

(55) Suppose K has a valuation. Let x, y be elements of K and x_1, y_1 be elements of ValuatRing v. If $x = x_1$ and $y = y_1$, then $x \cdot y = x_1 \cdot y_1$.

(56) If K has a valuation, then $0_{\mathrm{ValuatRing}\,v} = 0_K$.

(57) If K has a valuation, then $1_{\mathrm{ValuatRing}\,v} = 1_K$.

(58) If K has a valuation, then for every element x of K and for every element y of ValuatRing v such that $x = y$ holds $-x = -y$.

(59) If K has a valuation, then ValuatRing v is integral domain-like.

(60) If K has a valuation, then for every element y of ValuatRing v holds $\mathrm{power}_K(y, n) = \mathrm{power}_{\mathrm{ValuatRing}\,v}(y, n)$.

Let us consider K, v. Let us assume that K has a valuation. The functor PosElements v yields an ideal of ValuatRing v and is defined as follows:

(Def. 13) PosElements $v = \{x \in K \colon 0 < v(x)\}$.

Let us consider K, v. We introduce vp v as a synonym of PosElements v.

Next we state three propositions:

(61) If K has a valuation, then $a \in \mathrm{vp}\,v$ iff $0 < v(a)$.

(62) If K has a valuation, then $0_K \in \mathrm{vp}\,v$.

(63) If K has a valuation, then $1_K \notin \mathrm{vp}\,v$.

Let us consider K, v and let S be a non empty subset of K. Let us assume that K has a valuation and S is a subset of ValuatRing v. The functor $\min(S, v)$ yielding a subset of ValuatRing v is defined as follows:

(Def. 14) $\min(S, v) = v^{-1}(\{\inf(v^\circ S)\}) \cap S$.

The following four propositions are true:

(64) For every non empty subset S of K such that K has a valuation and S is a subset of ValuatRing v holds $\min(S, v) \subseteq S$.

(65) Let S be a non empty subset of K. Suppose K has a valuation and S is a subset of ValuatRing v. Let x be an element of K. Then $x \in \min(S, v)$ if and only if the following conditions are satisfied:

(i) $x \in S$, and

(ii) for every element y of K such that $y \in S$ holds $v(x) \leq v(y)$.

(66) Suppose K has a valuation. Let I be a non empty subset of K and x be an element of ValuatRing v. If I is an ideal of ValuatRing v and $x \in \min(I, v)$, then $I = \{x\}$–ideal.

(67) For every non empty double loop structure R holds every add closed non empty subset of R is a set closed w.r.t. the addition of R.

Let R be a ring and let P be a right ideal of R. A submodule of RightMod(R) is called a submodule of P if:

(Def. 15) The carrier of it $= P$.

Let R be a ring and let P be a right ideal of R. Note that there exists a submodule of P which is strict. Next we state the proposition

(68) Let R be a ring, P be an ideal of R, M be a submodule of P, a be a binary operation on P, z be an element of P, and m be a function from $P \times$ the carrier of R into P. Suppose $a =$ (the addition of $R) \restriction (P \times P)$ and $m =$ (the multiplication of $R) \restriction (P \times$ the carrier of $R)$ and $z =$ the zero of R. Then the right module structure of $M = \langle P, a, z, m \rangle$.

Let R be a ring, let M_1, M_2 be right modules over R, and let h be a function from M_1 into M_2. We say that h is scalar linear if and only if:

(Def. 16) For every element x of M_1 and for every element r of R holds $h(x \cdot r) = h(x) \cdot r$.

Let R be a ring, let M_1 be a right module over R, and let M_2 be a submodule of M_1. Observe that $\mathrm{incl}(M_2, M_1)$ is additive and scalar linear.

Next we state a number of propositions:

(69) If K has a valuation and b is an element of ValuatRing v, then $v(a) \leq v(a) + v(b)$.

(70) If K has a valuation and a is an element of ValuatRing v, then $\mathrm{power}_K(a, n)$ is an element of ValuatRing v.

(71) If K has a valuation, then for every element x of ValuatRing v such that $x \neq 0_K$ holds $\mathrm{power}_K(x, n) \neq 0_K$.

(72) If K has a valuation and $v(a) = 0$, then a is an element of ValuatRing v and a^{-1} is an element of ValuatRing v.

(73) If K has a valuation and $a \neq 0_K$ and a is an element of ValuatRing v and a^{-1} is an element of ValuatRing v, then $v(a) = 0$.

(74) If K has a valuation and $v(a) = 0$, then for every ideal I of ValuatRing v holds $a \in I$ iff $I =$ the carrier of ValuatRing v.

(75) If K has a valuation, then Pgenerator v is an element of ValuatRing v.

(76) If K has a valuation, then vp v is proper.

(77) If K has a valuation, then for every element x of ValuatRing v such that $x \notin$ vp v holds $v(x) = 0$.

(78) If K has a valuation, then vp v is prime.

(79) If K has a valuation, then for every proper ideal I of ValuatRing v holds $I \subseteq \mathrm{vp}\, v$.

(80) If K has a valuation, then $\mathrm{vp}\, v$ is maximal.

(81) If K has a valuation, then for every maximal ideal I of ValuatRing v holds $I = \mathrm{vp}\, v$.

(82) If K has a valuation, then NonNegElements normal-valuation v = NonNegElements v.

(83) If K has a valuation, then ValuatRing normal-valuation v = ValuatRing v.

References

[1] Emil Artin. *Algebraic Numbers and Algebraic Functions.* Gordon and Breach Science Publishers, 1994.
[2] Jonathan Backer, Piotr Rudnicki, and Christoph Schwarzweller. Ring ideals. *Formalized Mathematics*, 9(**3**):565–582, 2001.
[3] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[4] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[5] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[6] Józef Białas. Properties of fields. *Formalized Mathematics*, 1(**5**):807–812, 1990.
[7] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[8] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[9] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[10] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[11] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[12] Artur Korniłowicz. Quotient rings. *Formalized Mathematics*, 13(**4**):573–576, 2005.
[13] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.
[14] Michał Muzalewski. Construction of rings and left-, right-, and bi-modules over a ring. *Formalized Mathematics*, 2(**1**):3–11, 1991.
[15] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.
[16] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(**3**):501–505, 1990.
[17] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(**5**):821–827, 1990.
[18] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[19] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[20] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[21] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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