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Free Interpretation, Quotient Interpretation and Substitution of a Letter with a Term for First Order Languages[1]

Marco B. Caminati[2]
Mathematics Department "G.Castelnuovo"
Sapienza University of Rome
Piazzale Aldo Moro 5, 00185 Roma, Italy

Summary. Fourth of a series of articles laying down the bases for classical first order model theory. This paper supplies a toolkit of constructions to work with languages and interpretations, and results relating them. The free interpretation of a language, having as a universe the set of terms of the language itself, is defined.

The quotient of an interpreteation with respect to an equivalence relation is built, and shown to remain an interpretation when the relation respects it. Both the concepts of quotient and of respecting relation are defined in broadest terms, with respect to objects as general as possible.

Along with the trivial symbol substitution generally defined in [11], the more complex substitution of a letter with a term is defined, basing right on the free interpretation just introduced, which is a novel approach, to the author's knowledge. A first important result shown is that the quotient operation commute in some sense with term evaluation and reassignment functors, both introduced in [13] (theorem 3, theorem 15). A second result proved is substitution lemma (theorem 10, corresponding to III.8.3 of [15]). This will be vital for proving satisfiability theorem and correctness of a certain sequent derivation rule in [14]. A third result supplied is that if two given languages coincide on the letters of a given FinSequence, their evaluation of it will also coincide. This too will be instrumental in [14] for proving correctness of another rule. Also, the Depth functor is shown to be invariant with respect to term substitution in a formula.

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The notation and terminology used in this paper are introduced in the following articles: [1], [20], [17], [4], [5], [11], [12], [13], [19], [6], [7], [8], [16], [22], [2], [3], [9], [23], [25], [24], [18], [21], and [10].

For simplicity, we adopt the following rules: X, Y, x are sets, U, U_1, U_2 are non empty sets, u, u_1 are elements of U, R is a binary relation, f is a function, m, n are natural numbers, m_1, n_1 are elements of $\mathbb{N}$, S, S_1, S_2 are languages, s is an element of S, l, l_1, l_2 are literal elements of S, a is an of-atomic-formula element of S, r is a relational element of S, w is a string of S, t is a termal string of S, p_0 is a 0-w.f.f. string of S, p_1, p_2 are w.f.f. strings of S, I is an (S, U)-interpreter-like function, and t_1, t_0 are elements of AllTermsOf S.

Let us consider S, s and let V be an element of $((\text{AllSymbolsOf}\,S)^* \setminus \{\emptyset\})^*$. The functor s-compound V yields a string of S and is defined by:

(Def. 1) $\quad s\text{-compound}\,V = \langle s \rangle \frown S\text{-multiCat}(V)$.

Let us consider S, m_1, let s be a termal element of S, and let V be an $|\text{ar}\,s|$-element element of $S\text{-termsOfMaxDepth}(m_1)^*$. One can verify that s-compound V is $m_1 + 1$-termal.

Let us consider S, let s be a termal element of S, and let V be an $|\text{ar}\,s|$-element element of $(\text{AllTermsOf}\,S)^*$. Observe that s-compound V is termal.

Let us consider S, let s be a relational element of S, and let V be an $|\text{ar}\,s|$-element element of $(\text{AllTermsOf}\,S)^*$. One can check that s-compound V is 0-w.f.f..

Let us consider S, s. The functor s-compound yielding a function from $((\text{AllSymbolsOf}\,S)^* \setminus \{\emptyset\})^*$ into $(\text{AllSymbolsOf}\,S)^* \setminus \{\emptyset\}$ is defined by:

(Def. 2) $\quad$ For every element V of $((\text{AllSymbolsOf}\,S)^* \setminus \{\emptyset\})^*$ holds $s\text{-compound}(V) = s\text{-compound}\,V$.

Let us consider S and let s be a termal element of S.

Observe that $s\text{-compound}\restriction(\text{AllTermsOf}\,S)^{|\text{ar}\,s|}$ is AllTermsOf S-valued.

Let us consider S and let s be a relational element of S.

Note that $s\text{-compound}\restriction(\text{AllTermsOf}\,S)^{|\text{ar}\,s|}$ is AtomicFormulasOf S-valued.

Let us consider S, let s be an of-atomic-formula element of S, and let X be a set. The functor X-freeInterpreter s is defined as follows:

(Def. 3) $$X\text{-freeInterpreter}\,s = \begin{cases} s\text{-compound}\restriction(\text{AllTermsOf}\,S)^{|\text{ar}\,s|}, \\ \quad \text{if } s \text{ is not relational}, \\ (s\text{-compound}\restriction(\text{AllTermsOf}\,S)^{|\text{ar}\,s|}) \cdot \\ \quad (\chi_{X,\text{AtomicFormulasOf}\,S}\ \mathbf{qua}\ \text{binary relation}), \\ \quad \text{otherwise}. \end{cases}$$

Let us consider S, let s be an of-atomic-formula element of S, and let X be a set. Then X-freeInterpreter s is an interpreter of s and AllTermsOf S.

Let us consider S, X. The functor (S, X)-freeInterpreter yields a function and is defined as follows:

(Def. 4) $\operatorname{dom}((S,X)\text{-freeInterpreter}) = \text{OwnSymbolsOf}\, S$ and for every own element s of S holds $(S,X)\text{-freeInterpreter}(s) = X\text{-freeInterpreter}\, s$.

Let us consider S, X. Note that (S,X)-freeInterpreter is function yielding.

Let us consider S, X. Then (S,X)-freeInterpreter is an interpreter of S and AllTermsOf S.

Let us consider S, X. Note that (S,X)-freeInterpreter is $(S, \text{AllTermsOf}\, S)$-interpreter-like.

Then (S,X)-freeInterpreter is an element of AllTermsOf S-InterpretersOf S.

Let X, Y be non empty sets, let R be a relation between X and Y, and let n be a natural number. The functor n-placesOf R yielding a relation between X^n and Y^n is defined as follows:

(Def. 5) $n\text{-placesOf}\, R = \{\langle p, q\rangle; p$ ranges over elements of X^n, q ranges over elements of Y^n: $\bigwedge_{j\,:\,\mathrm{set}} (j \in \operatorname{Seg} n \;\Rightarrow\; \langle p(j), q(j)\rangle \in R)\}$.

Let X, Y be non empty sets, let R be a total relation between X and Y, and let n be a non zero natural number. Observe that n-placesOf R is total.

Let X, Y be non empty sets, let R be a total relation between X and Y, and let n be a natural number. Observe that n-placesOf R is total.

Let X, Y be non empty sets, let R be a relation between X and Y, and let n be a zero natural number. One can check that n-placesOf R is function-like.

Let X be a non empty set, let R be a binary relation on X, and let n be a natural number. The functor n-placesOf R yielding a binary relation on X^n is defined by:

(Def. 6) $n\text{-placesOf}\, R = n\text{-placesOf}(R$ **qua** relation between X and $X)$.

Let X be a non empty set, let R be a binary relation on X, and let n be a zero natural number. Then n-placesOf R is a binary relation on X^n and it can be characterized by the condition:

(Def. 7) $n\text{-placesOf}\, R = \{\langle \emptyset, \emptyset\rangle\}$.

Let X be a non empty set, let R be a symmetric total binary relation on X, and let us consider n. One can check that n-placesOf R is total.

Let X be a non empty set, let R be a symmetric total binary relation on X, and let us consider n. Observe that n-placesOf R is symmetric.

Let X be a non empty set, let R be a symmetric total binary relation on X, and let us consider n. Observe that n-placesOf R is symmetric and total.

Let X be a non empty set, let R be a transitive total binary relation on X, and let us consider n. Observe that n-placesOf R is transitive and total.

Let X be a non empty set, let R be an equivalence relation of X, and let us consider n. Observe that n-placesOf R is total, symmetric, and transitive.

Let X, Y be non empty sets, let E be an equivalence relation of X, let F be an equivalence relation of Y, and let R be a binary relation. The functor $R\,\text{quotient}(E,F)$ is defined by:

(Def. 8) R quotient$(E, F) = \{\langle e, f\rangle; e$ ranges over elements of Classes E, f ranges over elements of Classes $F : \bigvee_{x,y\,:\,\mathrm{set}} (x \in e \ \wedge\ y \in f \ \wedge\ \langle x, y\rangle \in R)\}$.

Let X, Y be non empty sets, let E be an equivalence relation of X, let F be an equivalence relation of Y, and let R be a binary relation. Then R quotient(E, F) is a relation between Classes E and Classes F.

Let E be a binary relation, let F be a binary relation, and let f be a function. We say that f is (E, F)-respecting if and only if:

(Def. 9) For all sets x_1, x_2 such that $\langle x_1, x_2\rangle \in E$ holds $\langle f(x_1), f(x_2)\rangle \in F$.

Let us consider S, U, let s be an of-atomic-formula element of S, let E be a binary relation on U, and let f be an interpreter of s and U. We say that f is E-respecting if and only if:

(Def. 10)(i) f is $(|\mathrm{ar}\, s|$-placesOf $E, E)$-respecting if s is not relational,

(ii) f is $(|\mathrm{ar}\, s|$-placesOf $E, \mathrm{id}_{Boolean})$-respecting, otherwise.

Let X, Y be non empty sets, let E be an equivalence relation of X, and let F be an equivalence relation of Y. Observe that there exists a function from X into Y which is (E, F)-respecting.

Let us consider S, U, let s be an of-atomic-formula element of S, and let E be an equivalence relation of U. Note that there exists an interpreter of s and U which is E-respecting.

Let X, Y be non empty sets, let E be an equivalence relation of X, and let F be an equivalence relation of Y. One can verify that there exists a function which is (E, F)-respecting.

Let X be a non empty set, let E be an equivalence relation of X, and let us consider n. Then n-placesOf E is an equivalence relation of X^n.

Let X be a non empty set and let x be an element of SmallestPartition(X). The functor DeTrivial x yielding an element of X is defined as follows:

(Def. 11) $x = \{\mathrm{DeTrivial}\, x\}$.

Let X be a non empty set. The functor peeler X yielding a function from $\{\{*\} : * \in X\}$ into X is defined as follows:

(Def. 12) For every element x of $\{\{*\} : * \in X\}$ holds $(\mathrm{peeler}\, X)(x) = \mathrm{DeTrivial}\, x$.

Let X be a non empty set and let E_1 be an equivalence relation of X. Note that every element of Classes E_1 is non empty.

Let X, Y be non empty sets, let E be an equivalence relation of X, let F be an equivalence relation of Y, and let f be an (E, F)-respecting function. One can check that f quotient(E, F) is function-like.

Let X, Y be non empty sets, let E be an equivalence relation of X, let F be an equivalence relation of Y, and let R be a total relation between X and Y. One can check that R quotient(E, F) is total.

Let X, Y be non empty sets, let E be an equivalence relation of X, let F be an equivalence relation of Y, and let f be an (E, F)-respecting function from X

into Y. Then f quotient(E, F) is a function from Classes E into Classes F.

Let X be a non empty set and let E be an equivalence relation of X. The functor E-class yields a function from X into Classes E and is defined by:

(Def. 13) For every element x of X holds E-class$(x) = \mathrm{EqClass}(E, x)$.

Let X be a non empty set and let E be an equivalence relation of X. Observe that E-class is onto.

Let X, Y be non empty sets. Note that there exists a relation between X and Y which is onto.

Let Y be a non empty set. Observe that there exists a Y-valued binary relation which is onto.

Let Y be a non empty set and let R be a Y-valued binary relation. Note that $R^{\smile}$ is Y-defined.

Let Y be a non empty set and let R be an onto Y-valued binary relation. Note that $R^{\smile}$ is total.

Let X, Y be non empty sets and let R be an onto relation between X and Y. One can check that $R^{\smile}$ is total.

Let Y be a non empty set and let R be an onto Y-valued binary relation. Note that $R^{\smile}$ is total.

Let us consider U, n and let E be an equivalence relation of U. The functor n-tuple2Class E yields a relation between $(\mathrm{Classes}\, E)^n$ and Classes(n-placesOf E) and is defined as follows:

(Def. 14) n-tuple2Class E = (n-placesOf(E-class **qua** relation between U and Classes $E)^{\smile}) \cdot (n$-placesOf E)-class.

Let us consider U, n and let E be an equivalence relation of U. Observe that n-tuple2Class E is function-like.

Let us consider U, n and let E be an equivalence relation of U. Note that n-tuple2Class E is total.

Let us consider U, n and let E be an equivalence relation of U. Then n-tuple2Class E is a function from $(\mathrm{Classes}\, E)^n$ into Classes(n-placesOf E).

Let us consider S, U, let s be an of-atomic-formula element of S, let E be an equivalence relation of U, and let f be an interpreter of s and U. The functor f quotient E is defined by:

(Def. 15)
$$f\,\mathrm{quotient}\,E = \begin{cases} (|\mathrm{ar}\,s|\text{-tuple2Class}\,E)\cdot \\ \quad (f\,\mathrm{quotient}(|\mathrm{ar}\,s|\text{-placesOf}\,E, E)), \\ \quad \text{if } s \text{ is not relational}, \\ (|\mathrm{ar}\,s|\text{-tuple2Class}\,E)\cdot \\ \quad (f\,\mathrm{quotient}(|\mathrm{ar}\,s|\text{-placesOf}\,E, \mathrm{id}_{\mathit{Boolean}}))\cdot \\ \quad \text{peeler}\,\mathit{Boolean}, \ \text{otherwise}. \end{cases}$$

Let us consider S, U, let s be an of-atomic-formula element of S, let E be an equivalence relation of U, and let f be an E-respecting interpreter of s and U. Then f quotient E is an interpreter of s and Classes E.

The following proposition is true

(1) Let X be a non empty set, E be an equivalence relation of X, and C_1, C_2 be elements of Classes E. If C_1 meets C_2, then $C_1 = C_2$.

Let us consider S. Observe that every element of OwnSymbolsOf S is own and every element of OwnSymbolsOf S is of-atomic-formula.

Let us consider S, U, let o be a non relational of-atomic-formula element of S, and let E be a binary relation on U. One can check that every interpreter of o and U which is E-respecting is also ($|\mathrm{ar}\, o|$-placesOf E, E)-respecting.

Let us consider S, U, let r be a relational element of S, and let E be a binary relation on U. Observe that every interpreter of r and U which is E-respecting is also ($|\mathrm{ar}\, r|$-placesOf $E, \mathrm{id}_{Boolean}$)-respecting.

Let us consider n, let U_1, U_2 be non empty sets, and let f be a function-like relation between U_1 and U_2. Note that n-placesOf f is function-like.

Let us consider U_1, U_2, let n be a zero natural number, and let R be a relation between U_1 and U_2. Note that (n-placesOf $R) \dot{-} \mathrm{id}_{\{\emptyset\}}$ is empty.

Let us consider X and let Y be a functional set. Observe that $X \cap Y$ is functional.

We now state the proposition

(2) For every element V of (AllTermsOf $S)^*$ there exists an element m_1 of $\mathbb{N}$ such that V is an element of S-termsOfMaxDepth$(m_1)^*$.

Let us consider S, U, let E be an equivalence relation of U, and let I be an (S, U)-interpreter-like function. We say that I is E-respecting if and only if:

(Def. 16) For every own element s of S holds $I(s)$ **qua** interpreter of s and U is E-respecting.

Let us consider S, U, let E be an equivalence relation of U, and let I be an (S, U)-interpreter-like function. The functor I quotient E yielding a function is defined as follows:

(Def. 17) $\mathrm{dom}(I\, \mathrm{quotient}\, E) =$ OwnSymbolsOf S and for every element o of OwnSymbolsOf S holds $(I\, \mathrm{quotient}\, E)(o) = I(o)\, \mathrm{quotient}\, E$.

Let us consider S, U, let E be an equivalence relation of U, and let I be an (S, U)-interpreter-like function. Then I quotient E can be characterized by the condition:

(Def. 18) $\mathrm{dom}(I\, \mathrm{quotient}\, E) =$ OwnSymbolsOf S and for every own element o of S holds $(I\, \mathrm{quotient}\, E)(o) = I(o)\, \mathrm{quotient}\, E$.

Let us consider S, U, let I be an (S, U)-interpreter-like function, and let E be an equivalence relation of U. Note that I quotient E is OwnSymbolsOf S-defined.

Let us consider S, U and let E be an equivalence relation of U. Note that there exists an element of U-InterpretersOf S which is E-respecting.

Let us consider S, U and let E be an equivalence relation of U. Observe that there exists an (S,U)-interpreter-like function which is E-respecting.

Let us consider S, U, let E be an equivalence relation of U, let o be an own element of S, and let I be an E-respecting (S,U)-interpreter-like function. One can check that $I(o)$ is E-respecting.

Let us consider S, U, let E be an equivalence relation of U, and let I be an E-respecting (S,U)-interpreter-like function. Observe that $I\,\mathrm{quotient}\,E$ is $(S, \mathrm{Classes}\,E)$-interpreter-like.

Let us consider S, U, let E be an equivalence relation of U, and let I be an E-respecting (S,U)-interpreter-like function. Then $I\,\mathrm{quotient}\,E$ is an element of $\mathrm{Classes}\,E$-InterpretersOf S.

The following propositions are true:

(3) Let E be an equivalence relation of U and I be an E-respecting (S,U)-interpreter-like function.
Then $(I\,\mathrm{quotient}\,E)$-TermEval $= E$-class $\cdot I$-TermEval .

(4) (S,X)-freeInterpreter-TermEval $= \mathrm{id}_{\mathrm{AllTermsOf}\,S}$.

(5) Let R be an equivalence relation of U_1, p_2 be a 0-w.f.f. string of S, and i be an R-respecting (S,U_1)-interpreter-like function. If S-firstChar$(p_2) \neq$ TheEqSymbOf S, then $(i\,\mathrm{quotient}\,R)$-AtomicEval $p_2 = i$-AtomicEval p_2.

Let us consider S, x, s, w. Then (x,s)-SymbolSubstIn w is a string of S.

Let us consider S, l_1, l_2, m and let t be an m-termal string of S. Note that (l_1,l_2)-SymbolSubstIn t is m-termal.

Let us consider S, t, l_1, l_2. One can check that (l_1,l_2)-SymbolSubstIn t is termal.

Let us consider S, l_1, l_2 and let p_2 be a 0-w.f.f. string of S. One can check that (l_1,l_2)-SymbolSubstIn p_2 is 0-w.f.f..

Let us consider S, let m_0 be a zero number, and let p_2 be an m_0-w.f.f. string of S. One can verify that Depth p_2 is zero.

Let us consider S, m, w. Then $w\,\mathrm{null}\,m$ is a string of S.

Let us consider S, p_2, m. Note that $p_2\,\mathrm{null}\,m$ is Depth $p_2 + m$-w.f.f..

Let us consider S, m and let p_2 be an m-w.f.f. string of S. Note that $m -$ Depth p_2 is non negative.

Let us consider S, l_1, l_2, m and let p_2 be an m-w.f.f. string of S. Observe that (l_1,l_2)-SymbolSubstIn p_2 is m-w.f.f..

Let us consider S, l_1, l_2, p_2. One can verify that (l_1,l_2)-SymbolSubstIn p_2 is w.f.f.. Observe that Depth$((l_1,l_2)$-SymbolSubstIn $p_2) \dot{-}$ Depth p_2 is empty.

The following proposition is true

(6) Let T be an $|\mathrm{ar}\,a|$-element element of $(\mathrm{AllTermsOf}\,S)^*$. Then

(i) if a is not relational, then $(X$-freeInterpreter $a)(T) = a$-compound T, and

(ii) if a is relational, then $(X$-freeInterpreter $a)(T) =$

$\chi_{X,\mathrm{AtomicFormulasOf}\,S}(a\text{-compound}\,T)$.

Let S be a language. One can verify that there exists a string of S which is termal and there exists a string of S which is 0-w.f.f..

One can prove the following proposition

(7) $(I\text{-TermEval} \cdot ((l, t_0)\,\mathrm{ReassignIn}(S, X)\text{-freeInterpreter}, t_0)\,\text{-TermEval}(n)){\restriction}$ $S\text{-termsOfMaxDepth}(n) =$ $((l, I\text{-TermEval}(t_0))\,\mathrm{ReassignIn}\,I, I\text{-TermEval}(t_0))\,\text{-TermEval}(n){\restriction}$ $S\text{-termsOfMaxDepth}(n)$.

Let us consider S, l, t_1, p_0. The functor $(l, t_1)\,\mathrm{AtomicSubst}\,p_0$ yielding a finite sequence is defined by:

(Def. 19) $(l, t_1)\,\mathrm{AtomicSubst}\,p_0 = \langle S\text{-firstChar}(p_0)\rangle{}^\frown S\text{-multiCat}(((l, t_1)\,\mathrm{ReassignIn}$ $(S, \emptyset)\text{-freeInterpreter})\text{-TermEval} \cdot \mathrm{SubTerms}\,p_0)$.

Let us consider S, l, t_1, p_0. Then $(l, t_1)\,\mathrm{AtomicSubst}\,p_0$ is a string of S.

Let us consider S, l, t_1, p_0. Observe that $(l, t_1)\,\mathrm{AtomicSubst}\,p_0$ is 0-w.f.f..

We now state the proposition

(8) $I\text{-AtomicEval}((l, t_1)\,\mathrm{AtomicSubst}\,p_0) =$ $((l, I\text{-TermEval}(t_1))\,\mathrm{ReassignIn}\,I)\text{-AtomicEval}\,p_0$.

Let us consider S, l_1, l_2, m. One can check that $(l_1\,\mathrm{SubstWith}\,l_2){\restriction}$ $S\text{-termsOfMaxDepth}(m)$ is $S\text{-termsOfMaxDepth}(m)$-valued.

Note that $(l_1\,\mathrm{SubstWith}\,l_2){\restriction}\,\mathrm{AllTermsOf}\,S$ is $\mathrm{AllTermsOf}\,S$-valued.

One can prove the following proposition

(9) If $l_2 \notin \mathrm{rng}\,p_1$, then for every element I of U-InterpretersOf S holds $((l_1, u_1)\,\mathrm{ReassignIn}\,I)\text{-TruthEval}\,p_1 =$ $((l_2, u_1)\,\mathrm{ReassignIn}\,I)\text{-TruthEval}((l_1, l_2)\,\text{-SymbolSubstIn}\,p_1)$.

Let us consider S, let us consider l, t, n, let f be a finite sequence-yielding function, and let us consider p_2. The functor $(l, t, n, f)\,\mathrm{Subst2}\,p_2$ yielding a finite sequence is defined by:

(Def. 20) $$(l, t, n, f)\,\mathrm{Subst2}\,p_2 = \begin{cases} \langle \mathrm{TheNorSymbOf}\,S\rangle \frown f(\mathrm{head}\,p_2) \frown f(\mathrm{tail}\,p_2), \\ \quad \text{if } \mathrm{Depth}\,p_2 = n+1 \text{ and } p_2 \text{ is not exal}, \\ \langle \text{the element of } \mathrm{LettersOf}\,S \setminus (\mathrm{rng}\,t \cup \mathrm{rng} \\ \quad \mathrm{head}\,p_2 \cup \{l\})\rangle \frown f((S\text{-firstChar}(p_2), \\ \quad \text{the element of } \mathrm{LettersOf}\,S \setminus (\mathrm{rng}\,t \cup \mathrm{rng} \\ \quad \mathrm{head}\,p_2 \cup \{l\}))\,\text{-SymbolSubstIn}\,\mathrm{head}\,p_2), \\ \quad \text{if } \mathrm{Depth}\,p_2 = n+1 \text{ and } p_2 \text{ is exal and} \\ \quad S\text{-firstChar}(p_2) \neq l, \\ f(p_2), \text{ otherwise.} \end{cases}$$

Let us consider S. One can verify that every element of $(\mathrm{AllFormulasOf}\,S)^{\mathrm{AllFormulasOf}\,S}$ is finite sequence-yielding.

Let us consider l, t, n, let f be an element of $(\mathrm{AllFormulasOf}\,S)^{\mathrm{AllFormulasOf}\,S}$, and let us consider p_2. Then $(l, t, n, f)\,\mathrm{Subst2}\,p_2$ is a w.f.f. string of S. Let f be an element of $(\text{AllFormulasOf } S)^{\text{AllFormulasOf } S}$, and let us consider p_2. Observe that $(l, t, n, f)\,\text{Subst2}\,p_2$ is w.f.f..

Let us consider n_1, let f be an element of $(\text{AllFormulasOf } S)^{\text{AllFormulasOf } S}$, and let us consider p_2. Then $(l, t, n_1, f)\,\text{Subst2}\,p_2$ is an element of AllFormulasOf S.

Let us consider S, l, t, n and let f be an element of $(\text{AllFormulasOf } S)^{\text{AllFormulasOf } S}$. The functor $(l, t, n, f)\,\text{Subst3}$ yields an element of $(\text{AllFormulasOf } S)^{\text{AllFormulasOf } S}$ and is defined as follows:

(Def. 21) For every p_2 holds $(l, t, n, f)\,\text{Subst3}(p_2) = (l, t, n, f)\,\text{Subst2}\,p_2$.

Let us consider S, l, t and let f be an element of $(\text{AllFormulasOf } S)^{\text{AllFormulasOf } S}$. The functor $(l, t)\,\text{Subst4}\,f$ yields a function from $\mathbb{N}$ into $(\text{AllFormulasOf } S)^{\text{AllFormulasOf } S}$ and is defined by:

(Def. 22) $((l, t)\,\text{Subst4}\,f)(0) = f$ and for every m holds $((l, t)\,\text{Subst4}\,f)(m + 1) = (l, t, m, ((l, t)\,\text{Subst4}\,f)(m))\,\text{Subst3}$.

Let us consider S, l, t. The functor $l\,\text{AtomicSubst}\,t$ yields a function from AtomicFormulasOf S into AtomicFormulasOf S and is defined by:

(Def. 23) For all p_0, t_1 such that $t_1 = t$ holds $(l\,\text{AtomicSubst}\,t)(p_0) = (l, t_1)\,\text{AtomicSubst}\,p_0$.

Let us consider S, l, t. The functor $l\,\text{Subst1}\,t$ yielding a function is defined as follows:

(Def. 24) $l\,\text{Subst1}\,t = \text{id}_{\text{AllFormulasOf } S} {+}\!\cdot (l\,\text{AtomicSubst}\,t)$.

Let us consider S, l, t. Then $l\,\text{Subst1}\,t$ is an element of $((\text{AllSymbolsOf } S)^*)^{\text{AllFormulasOf } S}$. Then $l\,\text{Subst1}\,t$ is an element of $(\text{AllFormulasOf } S)^{\text{AllFormulasOf } S}$.

Let us consider S, l, t, p_2. The functor $(l, t)\,\text{SubstIn}\,p_2$ yielding a w.f.f. string of S is defined as follows:

(Def. 25) $(l, t)\,\text{SubstIn}\,p_2 = ((l, t)\,\text{Subst4}(l\,\text{Subst1}\,t))(\text{Depth}\,p_2)(p_2)$.

Let us consider S, l, t, p_2. Note that $(l, t)\,\text{SubstIn}\,p_2$ is w.f.f..

One can prove the following proposition

(10) $\text{Depth}((l, t_1)\,\text{SubstIn}\,p_1) = \text{Depth}\,p_1$ and for every element I of U-InterpretersOf S holds $I\text{-TruthEval}((l, t_1)\,\text{SubstIn}\,p_1) = ((l, I\text{-TermEval}(t_1))\,\text{ReassignIn}\,I)\text{-TruthEval}\,p_1$.

Let us consider m, S, l, t and let p_2 be an m-w.f.f. string of S. Observe that $(l, t)\,\text{SubstIn}\,p_2$ is m-w.f.f..

The following propositions are true:

(11) Let I_1 be an element of U-InterpretersOf S_1 and I_2 be an element of U-InterpretersOf S_2. Suppose $I_1 \restriction X = I_2 \restriction X$ and (the adicity of $S_1) \restriction X =$ (the adicity of $S_2) \restriction X$. Then $I_1\text{-TermEval} \restriction X^* = I_2\text{-TermEval} \restriction X^*$.

(12) Suppose TheNorSymbOf S_1 = TheNorSymbOf S_2 and TheEqSymbOf S_1 = TheEqSymbOf S_2 and (the adicity of S_1)$\restriction$OwnSymbolsOf S_1 = (the adicity of S_2)$\restriction$OwnSymbolsOf S_1. Let I_1 be an element of U-InterpretersOf S_1, I_2 be an element of U-InterpretersOf S_2, and p_4 be a w.f.f. string of S_1. Suppose $I_1\restriction$OwnSymbolsOf $S_1 = I_2\restriction$OwnSymbolsOf S_1. Then there exists a w.f.f. string p_3 of S_2 such that $p_3 = p_4$ and I_2-TruthEval p_3 = I_1-TruthEval p_4.

(13) For all elements I_1, I_2 of U-InterpretersOf S such that $I_1\restriction$(rng $p_2 \cap$ OwnSymbolsOf S) = $I_2\restriction$(rng $p_2\cap$OwnSymbolsOf S) holds I_1-TruthEval p_2 = I_2-TruthEval p_2.

(14) For every element I of U-InterpretersOf S such that l is X-absent and X is I-satisfied holds X is (l, u) ReassignIn I-satisfied.

(15) For every equivalence relation E of U and for every E-respecting element i of U-InterpretersOf S holds $(l, E\text{-class}(u))$ ReassignIn(i quotient E) = $((l, u)$ ReassignIn $i)$ quotient E.

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