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First Order Languages: Further Syntax and Semantics[1]

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Summary. Third of a series of articles laying down the bases for classical first order model theory. Interpretation of a language in a universe set. Evaluation of a term in a universe. Truth evaluation of an atomic formula. Reassigning the value of a symbol in a given interpretation. Syntax and semantics of a non atomic formula are then defined concurrently (this point is explained in [16], 4.2.1). As a consequence, the evaluation of any w.f.f. string and the relation of logical implication are introduced. Depth of a formula. Definition of satisfaction and entailment (aka entailment or logical implication) relations, see [18] III.3.2 and III.4.1 respectively.

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The terminology and notation used in this paper have been introduced in the following papers: [7], [1], [23], [6], [8], [17], [14], [15], [22], [9], [10], [11], [2], [21], [26], [24], [5], [3], [4], [12], [27], [28], [19], [20], [25], and [13].

For simplicity, we follow the rules: m, n denote natural numbers, m_1 denotes an element of $\mathbb{N}$, A, B, X, Y, Z, x, y denote sets, S, S_1, S_2 denote languages, s denotes an element of S, w, w_1, w_2 denote strings of S, U denotes a non empty set, f, g denote functions, and p, p_2 denote finite sequences.

Let us consider S. Then TheNorSymbOf S is an element of S.

Let U be a non empty set. The functor U-deltaInterpreter yielding a function from U^2 into *Boolean* is defined by:

(Def. 1) U-deltaInterpreter $= \chi_{(\text{the concatenation of } U)^{\circ}(\mathrm{id}_{U^1}), U^2}$.

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Let X be a set. Then id_X is an equivalence relation of X.

Let S be a language, let U be a non empty set, and let s be an of-atomic-formula element of S. Interpreter of s and U is defined as follows:

(Def. 2)(i) It is a function from $U^{|\mathrm{ar}\, s|}$ into *Boolean* if s is relational,

(ii) it is a function from $U^{|\mathrm{ar}\, s|}$ into U, otherwise.

Let us consider S, U and let s be an of-atomic-formula element of S. We see that the interpreter of s and U is a function from $U^{|\mathrm{ar}\, s|}$ into $U \cup \mathit{Boolean}$.

Let us consider S, U and let s be a termal element of S. One can verify that every interpreter of s and U is U-valued.

Let S be a language. Note that every element of S which is literal is also own.

Let us consider S, U. A function is called an interpreter of S and U if:

(Def. 3) For every own element s of S holds $\mathrm{it}(s)$ is an interpreter of s and U.

Let us consider S, U, f. We say that f is (S,U)-interpreter-like if and only if:

(Def. 4) f is an interpreter of S and U and function yielding.

Let us consider S and let U be a non empty set. One can verify that every function which is (S,U)-interpreter-like is also function yielding.

Let us consider S, U and let s be an own element of S. Observe that every interpreter of s and U is non empty.

Let S be a language and let U be a non empty set. Note that there exists a function which is (S,U)-interpreter-like.

Let us consider S, U, let I be an (S,U)-interpreter-like function, and let s be an own element of S. Then $I(s)$ is an interpreter of s and U.

Let S be a language, let U be a non empty set, let I be an (S,U)-interpreter-like function, let x be an own element of S, and let f be an interpreter of x and U. One can check that $I {+\cdot} (x \dot{\longmapsto} f)$ is (S,U)-interpreter-like.

Let us consider f, x, y. The functor $(x,y)\, \mathrm{ReassignIn}\, f$ yields a function and is defined by:

(Def. 5) $(x,y)\, \mathrm{ReassignIn}\, f = f {+\cdot} (x \dot{\longmapsto} (\emptyset \dot{\longmapsto} y))$.

Let S be a language, let U be a non empty set, let I be an (S,U)-interpreter-like function, let x be a literal element of S, and let u be an element of U. One can verify that $(x,u)\, \mathrm{ReassignIn}\, I$ is (S,U)-interpreter-like.

Let S be a language. One can check that $\mathrm{AllSymbolsOf}\, S$ is non empty.

Let Y be a set and let X, Z be non empty sets. Observe that every function from X into Z^Y is function yielding.

Let X, Y, Z be non empty sets. One can verify that there exists a function from X into Z^Y which is function yielding.

Let f be a function yielding function and let g be a function. The functor $[g, f]$ yields a function and is defined by:

(Def. 6) $\mathrm{dom}[g, f] = \mathrm{dom}\, f$ and for every x such that $x \in \mathrm{dom}\, f$ holds $[g, f](x) = g \cdot f(x)$.

Let f be an empty function and let g be a function. One can verify that $[g, f]$ is empty.

Let f be a function yielding function and let g be a function. The functor $[f, g]$ yielding a function is defined as follows:

(Def. 7) $\mathrm{dom}[f, g] = \mathrm{dom}\, f \cap \mathrm{dom}\, g$ and for every set x such that $x \in \mathrm{dom}[f, g]$ holds $[f, g](x) = f(x)(g(x))$.

Let f be a function yielding function and let g be an empty function. One can check that $[f, g]$ is empty.

Let X be a finite sequence-membered set. Observe that every function which is X-valued is also function yielding.

Let E, D be non empty sets, let p be a D-valued finite sequence, and let h be a function from D into E. Note that $h \cdot p$ is $\mathrm{len}\, p$-element.

Let X, Y be non empty sets, let f be a function from X into Y, and let p be an X-valued finite sequence. One can verify that $f \cdot p$ is finite sequence-like.

Let E, D be non empty sets, let n be a natural number, let p be an n-element D-valued finite sequence, and let h be a function from D into E. Observe that $h \cdot p$ is n-element.

We now state the proposition

(1) For every 0-termal string t_0 of S holds $t_0 = \langle S\text{-firstChar}(t_0)\rangle$.

Let us consider S, let U be a non empty set, let u be an element of U, and let I be an (S, U)-interpreter-like function. The functor (I, u)-TermEval yields a function from $\mathbb{N}$ into $U^{\mathrm{AllTermsOf}\, S}$ and is defined as follows:

(Def. 8) $(I, u)\text{-TermEval}(0) = \mathrm{AllTermsOf}\, S \longmapsto u$ and for every m_1 holds $(I, u)\text{-TermEval}(m_1 + 1) = [I \cdot S\text{-firstChar}, [((I, u)\text{-TermEval}(m_1)$ **qua** function$), S\text{-subTerms}]]$.

Let us consider S, U, let I be an (S, U)-interpreter-like function, and let t be an element of AllTermsOf S. The functor I-TermEval t yields an element of U and is defined as follows:

(Def. 9) For every element u_1 of U and for every m_1 such that $t \in S\text{-termsOfMaxDepth}(m_1)$ holds $I\text{-TermEval}\, t = (I, u_1)\text{-TermEval}(m_1 + 1)(t)$.

Let us consider S, U and let I be an (S, U)-interpreter-like function. The functor I-TermEval yielding a function from AllTermsOf S into U is defined by:

(Def. 10) For every element t of AllTermsOf S holds $I\text{-TermEval}(t) = I\text{-TermEval}\, t$.

Let us consider S, U and let I be an (S, U)-interpreter-like function. The functor $I ===$ yielding a function is defined as follows:

(Def. 11) $I ==== I {+\cdot} (\mathrm{TheEqSymbOf}\, S \dot{\longmapsto} U\text{-deltaInterpreter})$.

Let us consider S, U, let I be an (S,U)-interpreter-like function, and let x be a set. We say that x is I-extension if and only if:

(Def. 12) $x = I ===$.

Let us consider S, U and let I be an (S,U)-interpreter-like function. Note that $I ===$ is I-extension and every set which is I-extension is also function-like. Observe that there exists a function which is I-extension. Observe that $I ===$ is (S,U)-interpreter-like.

Let f be an I-extension function, and let s be an of-atomic-formula element of S. Then $f(s)$ is an interpreter of s and U.

Let p_1 be a 0-w.f.f. string of S. The functor I-AtomicEval p_1 is defined as follows:

(Def. 13) $I\text{-AtomicEval}\, p_1 = (I === (S\text{-firstChar}(p_1)))(I\text{-TermEval} \cdot \text{SubTerms}\, p_1)$.

Let us consider S, U, let I be an (S,U)-interpreter-like function, and let p_1 be a 0-w.f.f. string of S. Then I-AtomicEval p_1 is an element of *Boolean*. Note that $I \restriction \text{OwnSymbolsOf}\, S$ is $(U^* \dot{\rightarrow} (U \cup \mathit{Boolean}))$-valued and $I \restriction \text{OwnSymbolsOf}\, S$ is (S,U)-interpreter-like.

Let us consider S, U and let I be an (S,U)-interpreter-like function. Observe that $I \restriction \text{OwnSymbolsOf}\, S$ is total.

Let us consider S, U. The functor U-InterpretersOf S is defined by:

(Def. 14) $U\text{-InterpretersOf}\, S = \{f \in (U^* \dot{\rightarrow} (U \cup \mathit{Boolean}))^{\text{OwnSymbolsOf}\, S} \colon f$ is (S,U)-interpreter-like$\}$.

Let us consider S, U. Then U-InterpretersOf S is a subset of $(U^* \dot{\rightarrow} (U \cup \mathit{Boolean}))^{\text{OwnSymbolsOf}\, S}$. Observe that U-InterpretersOf S is non empty. One can verify that every element of U-InterpretersOf S is (S,U)-interpreter-like. The functor S-TruthEval U yields a function from $(U\text{-InterpretersOf}\, S) \times \text{AtomicFormulasOf}\, S$ into *Boolean* and is defined by:

(Def. 15) For every element I of U-InterpretersOf S and for every element p_1 of AtomicFormulasOf S holds $(S\text{-TruthEval}\, U)(I, p_1) = I\text{-AtomicEval}\, p_1$.

Let us consider S, U, let I be an element of U-InterpretersOf S, let f be a partial function from $(U\text{-InterpretersOf}\, S) \times ((\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\})$ to *Boolean*, and let p_1 be an element of $(\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\}$. The functor f-ExFunctor(I,p_1) yielding an element of *Boolean* is defined as follows:

(Def. 16) $$f\text{-ExFunctor}(I,p_1) = \begin{cases} \mathit{true}, & \text{if there exists an element } u \text{ of } U \text{ and there exists a literal element } v \text{ of } S \text{ such that } p_1(1) = v \text{ and } f((v,u)\,\text{ReassignIn}\, I,\ (p_1)_{\lfloor 1}) = \mathit{true}, \\ \mathit{false}, & \text{otherwise.} \end{cases}$$

Let us consider S, U and let g be an element of $(U\text{-InterpretersOf}\, S) \times ((\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\}) \dot{\rightarrow} \mathit{Boolean}$. The functor ExIterator g yields a partial function from $(U\text{-InterpretersOf}\, S) \times ((\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\})$ to *Boolean* and is defined by the conditions (Def. 17).

(Def. 17)(i) For every element x of U-InterpretersOf S and for every element y of $(\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\}$ holds $\langle x, y\rangle \in \text{dom}\,\text{ExIterator}\, g$ iff there exists a literal element v of S and there exists a string w of S such that $\langle x, w\rangle \in \text{dom}\, g$ and $y = \langle v\rangle \frown w$, and

(ii) for every element x of U-InterpretersOf S and for every element y of $(\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\}$ such that $\langle x, y\rangle \in \text{dom}\,\text{ExIterator}\, g$ holds $(\text{ExIterator}\, g)(x, y) = g\text{-ExFunctor}(x, y)$.

Let us consider S, U, let f be a partial function from $(U\text{-InterpretersOf}\, S) \times ((\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\})$ to *Boolean*, let I be an element of U-InterpretersOf S, and let p_1 be an element of $(\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\}$.

The functor f-NorFunctor(I, p_1) yielding an element of *Boolean* is defined by:

(Def. 18) $$f\text{-NorFunctor}(I, p_1) = \begin{cases} \mathit{true}, & \text{if there exist elements } w_1,\, w_2 \text{ of } (\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\} \text{ such that } \langle I, w_1\rangle \in \text{dom}\, f \text{ and } f(I, w_1) = \mathit{false} \text{ and } f(I, w_2) = \mathit{false} \text{ and } p_1 = \langle \text{TheNorSymbOf}\, S\rangle \frown w_1 \frown w_2, \\ \mathit{false}, & \text{otherwise.} \end{cases}$$

Let us consider S, U and let g be an element of $(U\text{-InterpretersOf}\, S) \times ((\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\}) \dot{\rightarrow} \mathit{Boolean}$. The functor NorIterator g yielding a partial function from $(U\text{-InterpretersOf}\, S) \times ((\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\})$ to *Boolean* is defined by the conditions (Def. 19).

(Def. 19)(i) For every element x of U-InterpretersOf S and for every element y of $(\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\}$ holds $\langle x, y\rangle \in \text{dom}\,\text{NorIterator}\, g$ iff there exist elements p_3, p_4 of $(\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\}$ such that $y = \langle \text{TheNorSymbOf}\, S\rangle \frown p_3 \frown p_4$ and $\langle x, p_3\rangle$, $\langle x, p_4\rangle \in \text{dom}\, g$, and

(ii) for every element x of U-InterpretersOf S and for every element y of $(\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\}$ such that $\langle x, y\rangle \in \text{dom}\,\text{NorIterator}\, g$ holds $(\text{NorIterator}\, g)(x, y) = g\text{-NorFunctor}(x, y)$.

Let us consider S, U. The functor (S, U)-TruthEval yields a function from $\mathbb{N}$ into $(U\text{-InterpretersOf}\, S) \times ((\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\}) \dot{\rightarrow} \mathit{Boolean}$ and is defined as follows:

(Def. 20) $(S, U)\text{-TruthEval}(0) = S\text{-TruthEval}\, U$ and for every m_1 holds $(S, U)\text{-TruthEval}(m_1{+}1) = \text{ExIterator}(S, U)\text{-TruthEval}(m_1) {+\cdot}\, \text{NorIterator}\, (S, U)\text{-TruthEval}(m_1) {+\cdot} (S, U)\text{-TruthEval}(m_1)$.

Next we state the proposition

(2) For every (S, U)-interpreter-like function I holds $I \restriction \text{OwnSymbolsOf}\, S \in U\text{-InterpretersOf}\, S$.

Let S be a language, let m be a natural number, and let U be a non empty set.

The functor (S, U)-TruthEval m yielding an element of (U-InterpretersOf S) $\times$ ((AllSymbolsOf $S)^* \setminus \{\emptyset\}) \dot{\rightarrow}$ *Boolean* is defined as follows:

(Def. 21) For every m_1 such that $m = m_1$ holds (S, U)-TruthEval m = (S, U)-TruthEval(m_1).

Let us consider S, U, m and let I be an element of U-InterpretersOf S. The functor (I, m)-TruthEval yields an element of ((AllSymbolsOf $S)^* \setminus \{\emptyset\}) \dot{\rightarrow}$ *Boolean* and is defined by:

(Def. 22) (I, m)-TruthEval = (curry((S, U)-TruthEval m))(I).

Let us consider S, m. The functor S-formulasOfMaxDepth m yielding a subset of (AllSymbolsOf $S)^* \setminus \{\emptyset\}$ is defined as follows:

(Def. 23) For every non empty set U and for every element I of U-InterpretersOf S and for every element m_1 of $\mathbb{N}$ such that $m = m_1$ holds S-formulasOfMaxDepth m = dom((I, m_1)-TruthEval).

Let us consider S, m, w. We say that w is m-w.f.f. if and only if:

(Def. 24) $w \in S$-formulasOfMaxDepth m.

Let us consider S, w. We say that w is w.f.f. if and only if:

(Def. 25) There exists m such that w is m-w.f.f..

Let us consider S. Note that every string of S which is 0-w.f.f. is also 0-w.f.f. and every string of S which is 0-w.f.f. is also 0-w.f.f.. Let us consider m. One can check that every string of S which is m-w.f.f. is also w.f.f.. Let us consider n. One can check that every string of S which is $m + 0 \cdot n$-w.f.f. is also $m + n$-w.f.f..

Let us consider S, m. Observe that there exists a string of S which is m-w.f.f.. Note that S-formulasOfMaxDepth m is non empty. One can verify that there exists a string of S which is w.f.f..

Let us consider S, U, let I be an element of U-InterpretersOf S, and let w be a w.f.f. string of S. The functor I-TruthEval w yields an element of *Boolean* and is defined as follows:

(Def. 26) For every natural number m such that w is m-w.f.f. holds I-TruthEval w = (I, m)-TruthEval(w).

Let us consider S. The functor AllFormulasOf S is defined by:

(Def. 27) AllFormulasOf S = $\{w; w$ ranges over strings of S: $\bigvee_m$ w is m-w.f.f.$\}$.

Let us consider S. One can check that AllFormulasOf S is non empty.

For simplicity, we follow the rules: u, u_1, u_2 are elements of U, t is a termal string of S, I is an (S, U)-interpreter-like function, l, l_1, l_2 are literal elements of S, m_2, n_1 are non zero natural numbers, p_0 is a 0-w.f.f. string of S, and p_5, p_1, p_3, p_4 are w.f.f. strings of S.

The following propositions are true:

(3) (I, u)-TermEval$(m + 1)(t)$ = $I(S$-firstChar$(t))((I, u)$-TermEval(m) $\cdot$ SubTerms t) and if t is 0-termal, then (I, u)-TermEval$(m + 1)(t)$ = $I(S$-firstChar$(t))(\emptyset)$.

(4) For every m-termal string t of S holds (I, u_1)-TermEval$(m+1)(t) =$ (I, u_2)-TermEval$(m+1+n)(t)$.

(5) curry((S, U)-TruthEval m) is a function from U-InterpretersOf S into $((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}) \dot{\to} Boolean$.

(6) $x \in X \cup Y \cup Z$ iff $x \in X$ or $x \in Y$ or $x \in Z$.

(7) S-formulasOfMaxDepth $0 =$ AtomicFormulasOf S.

Let us consider S, m. Then S-formulasOfMaxDepth m can be characterized by the condition:

(Def. 28) For every non empty set U and for every element I of U-InterpretersOf S holds S-formulasOfMaxDepth $m = \text{dom}((I, m)\text{-TruthEval})$.

Next we state the proposition

(8) (S, U)-TruthEval $m \in Boolean^{(U\text{-InterpretersOf } S) \times (S\text{-formulasOfMaxDepth } m)}$ and (S, U)-TruthEval$(m) \in Boolean^{(U\text{-InterpretersOf } S) \times (S\text{-formulasOfMaxDepth } m)}$.

Let us consider S, m. The functor m-ExFormulasOf S is defined by:

(Def. 29) m-ExFormulasOf $S = \{\langle v \rangle ^\frown p_1 : v$ ranges over elements of LettersOf S, p_1 ranges over elements of S-formulasOfMaxDepth $m\}$.

The functor m-NorFormulasOf S is defined as follows:

(Def. 30) m-NorFormulasOf $S = \{\langle$TheNorSymbOf $S\rangle ^\frown p_3 ^\frown p_4 : p_3$ ranges over elements of S-formulasOfMaxDepth m, p_4 ranges over elements of S-formulasOfMaxDepth $m\}$.

Let us consider S and let w_1, w_2 be strings of S. Then $w_1 ^\frown w_2$ is a string of S.

Let us consider S, s. Then $\langle s \rangle$ is a string of S.

One can prove the following two propositions:

(9) S-formulasOfMaxDepth$(m+1) =$ (m-ExFormulasOf S)$\cup$(m-NorFormulasOf S)$\cup$(S-formulasOfMaxDepth m).

(10) AtomicFormulasOf S is S-prefix.

Let us consider S. Note that AtomicFormulasOf S is S-prefix. Observe that S-formulasOfMaxDepth 0 is S-prefix.

Let us consider p_1. The functor Depth p_1 yielding a natural number is defined by:

(Def. 31) p_1 is Depth p_1-w.f.f. and for every n such that p_1 is n-w.f.f. holds Depth $p_1 \leq n$.

Let us consider S, m and let p_3, p_4 be m-w.f.f. strings of S. Note that $\langle$TheNorSymbOf $S\rangle ^\frown p_3 ^\frown p_4$ is $m+1$-w.f.f..

Let us consider S, p_3, p_4. Observe that $\langle$TheNorSymbOf $S\rangle ^\frown p_3 ^\frown p_4$ is w.f.f..

Let us consider S, m, let p_1 be an m-w.f.f. string of S, and let v be a literal element of S. Note that $\langle v \rangle ^\frown p_1$ is $m+1$-w.f.f..

Let us consider S, l, p_1. Note that $\langle l \rangle \mathbin{^\frown} p_1$ is w.f.f..

Let us consider S, w and let s be a non relational element of S. One can check that $\langle s \rangle \mathbin{^\frown} w$ is non 0-w.f.f..

Let us consider S, w_1, w_2 and let s be a non relational element of S. Observe that $\langle s \rangle \mathbin{^\frown} w_1 \mathbin{^\frown} w_2$ is non 0-w.f.f..

Let us consider S. Observe that TheNorSymbOf S is non relational.

Let us consider S, w. Observe that $\langle \text{TheNorSymbOf}\, S \rangle \mathbin{^\frown} w$ is non 0-w.f.f..

Let us consider S, l, w. Note that $\langle l \rangle \mathbin{^\frown} w$ is non 0-w.f.f..

Let us consider S, w. We say that w is exal if and only if:

(Def. 32) S-firstChar(w) is literal.

Let us consider S, w, l. One can verify that $\langle l \rangle \mathbin{^\frown} w$ is exal.

Let us consider S, m_2. Observe that there exists an m_2-w.f.f. string of S which is exal.

Let us consider S. Note that every string of S which is exal is also non 0-w.f.f..

Let us consider S, m_2. One can check that there exists an exal m_2-w.f.f. string of S which is non 0-w.f.f..

Let us consider S. One can verify that there exists an exal w.f.f. string of S which is non 0-w.f.f..

Let us consider S and let p_1 be a non 0-w.f.f. w.f.f. string of S. Note that Depth p_1 is non zero.

Let us consider S and let w be a non 0-w.f.f. w.f.f. string of S. Observe that S-firstChar(w) is non relational.

Let us consider S, m. Observe that S-formulasOfMaxDepth m is S-prefix. Then AllFormulasOf S is a subset of $(\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\}$. Observe that every element of AllFormulasOf S is w.f.f.. Note that AllFormulasOf S is S-prefix.

We now state three propositions:

(11) $\operatorname{dom} \text{NorIterator}((S,U)\text{-TruthEval}\, m) = (U\text{-InterpretersOf}\, S) \times (m\text{-NorFormulasOf}\, S)$.

(12) $\operatorname{dom} \text{ExIterator}((S,U)\text{-TruthEval}\, m) = (U\text{-InterpretersOf}\, S) \times (m\text{-ExFormulasOf}\, S)$.

(13) $U\text{-deltaInterpreter}^{-1}(\{1\}) = \{\langle u, u \rangle : u$ ranges over elements of $U\}$.

Let us consider S. Then TheEqSymbOf S is an element of S.

Let us consider S. One can verify that $\text{ar}\, \text{TheEqSymbOf}\, S + 2$ is zero and $|\text{ar}\, \text{TheEqSymbOf}\, S| - 2$ is zero.

We now state two propositions:

(14) Let p_1 be a 0-w.f.f. string of S and I be an (S,U)-interpreter-like function. Then

(i) if S-firstChar$(p_1) \neq$ TheEqSymbOf S, then $I\text{-AtomicEval}\, p_1 = I(S\text{-firstChar}(p_1))(I\text{-TermEval} \cdot \text{SubTerms}\, p_1)$, and

(ii) if S-firstChar(p_1) = TheEqSymbOf S, then I-AtomicEval p_1 = U-deltaInterpreter(I-TermEval · SubTerms p_1).

(15) Let I be an (S, U)-interpreter-like function and p_1 be a 0-w.f.f. string of S. If S-firstChar(p_1) = TheEqSymbOf S, then I-AtomicEval $p_1 = 1$ iff I-TermEval((SubTerms p_1)(1)) = I-TermEval((SubTerms p_1)(2)).

Let us consider S, m. One can check that m-ExFormulasOf S is non empty. Note that m-NorFormulasOf S is non empty. Then m-NorFormulasOf S is a subset of $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$.

Let us consider S and let w be an exal string of S. One can verify that S-firstChar(w) is literal.

Let us consider S, m. Observe that every element of m-NorFormulasOf S is non exal. Then m-ExFormulasOf S is a subset of $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$.

Let us consider S, m. One can check that every element of m-ExFormulasOf S is exal.

Let us consider S. One can check that there exists an element of S which is non literal.

Let us consider S, w and let s be a non literal element of S. Note that $\langle s \rangle \frown w$ is non exal.

Let us consider S, w_1, w_2 and let s be a non literal element of S. Observe that $\langle s \rangle \frown w_1 \frown w_2$ is non exal.

Let us consider S. Note that TheNorSymbOf S is non literal.

Next we state the proposition

(16) $p_1 \in$ AllFormulasOf S.

Let us consider S, m, w. We introduce w is m-non-w.f.f. as an antonym of w is m-w.f.f..

Let us consider m, S. One can verify that every string of S which is non m-w.f.f. is also m-non-w.f.f..

Let us consider S, p_3, p_4. Observe that $\langle \text{TheNorSymbOf } S \rangle \frown p_3 \frown p_4$ is $\max(\text{Depth } p_3, \text{Depth } p_4)$-non-w.f.f..

Let us consider S, p_1, l. Note that $\langle l \rangle \frown p_1$ is Depth p_1-non-w.f.f..

Let us consider S, p_1, l. One can check that $\langle l \rangle \frown p_1$ is $1 + \text{Depth } p_1$-w.f.f..

Let us consider S, U. Observe that every element of U-InterpretersOf S is OwnSymbolsOf S-defined.

Let us consider S, U. Note that there exists an element of U-InterpretersOf S which is OwnSymbolsOf S-defined.

Let us consider S, U. Note that every OwnSymbolsOf S-defined element of U-InterpretersOf S is total.

Let us consider S, U, let I be an element of U-InterpretersOf S, let x be a literal element of S, and let u be an element of U. Then (x, u) ReassignIn I is an element of U-InterpretersOf S.

In the sequel I denotes an element of U-InterpretersOf S.

Let us consider S, w. The functor $\operatorname{xnot} w$ yields a string of S and is defined as follows:

(Def. 33) $\operatorname{xnot} w = \langle \text{TheNorSymbOf}\, S \rangle \frown w \frown w$.

Let us consider S, m and let p_1 be an m-w.f.f. string of S. Observe that $\operatorname{xnot} p_1$ is $m+1$-w.f.f..

Let us consider S, p_1. Note that $\operatorname{xnot} p_1$ is w.f.f..

Let us consider S. One can verify that $\text{TheEqSymbOf}\, S$ is non own.

Let us consider S, X. We say that X is S-mincover if and only if:

(Def. 34) For every p_1 holds $p_1 \in X$ iff $\operatorname{xnot} p_1 \notin X$.

One can prove the following propositions:

(17) $\text{Depth}(\langle \text{TheNorSymbOf}\, S \rangle \frown p_3 \frown p_4) = 1 + \max(\text{Depth}\, p_3, \text{Depth}\, p_4)$ and $\text{Depth}(\langle l \rangle \frown p_3) = \text{Depth}\, p_3 + 1$.

(18) If $\text{Depth}\, p_1 = m+1$, then p_1 is exal iff $p_1 \in m\text{-ExFormulasOf}\, S$ and p_1 is non exal iff $p_1 \in m\text{-NorFormulasOf}\, S$.

(19) $I\text{-TruthEval}\langle l \rangle \frown p_1 = \mathit{true}$ iff there exists u such that $((l,u)\, \text{ReassignIn}\, I)\text{-TruthEval}\, p_1 = 1$ and $I\text{-TruthEval}\langle \text{TheNorSymbOf}\, S \rangle \frown p_3 \frown p_4 = \mathit{true}$ iff $I\text{-TruthEval}\, p_3 = \mathit{false}$ and $I\text{-TruthEval}\, p_4 = \mathit{false}$.

In the sequel I denotes an (S,U)-interpreter-like function.

One can prove the following two propositions:

(20) $(I,u)\text{-TermEval}(m+1) \restriction S\text{-termsOfMaxDepth}(m) = I\text{-TermEval} \restriction S\text{-termsOfMaxDepth}(m)$.

(21) $I\text{-TermEval}(t) = I(S\text{-firstChar}(t))(I\text{-TermEval} \cdot \text{SubTerms}\, t)$.

Let us consider S, p_1. The functor $\text{SubWffsOf}\, p_1$ is defined as follows:

(Def. 35)(i) There exist p_3, p such that p is $\text{AllSymbolsOf}\, S$-valued and $\text{SubWffsOf}\, p_1 = \langle\!\langle p_3, p \rangle\!\rangle$ and $p_1 = \langle S\text{-firstChar}(p_1) \rangle \frown p_3 \frown p$ if p_1 is non 0-w.f.f.,

(ii) $\text{SubWffsOf}\, p_1 = \langle\!\langle p_1, \emptyset \rangle\!\rangle$, otherwise.

Let us consider S, p_1. The functor $\text{head}\, p_1$ yields a w.f.f. string of S and is defined as follows:

(Def. 36) $\text{head}\, p_1 = (\text{SubWffsOf}\, p_1)_{\mathbf{1}}$.

The functor $\text{tail}\, p_1$ yields an element of $(\text{AllSymbolsOf}\, S)^*$ and is defined by:

(Def. 37) $\text{tail}\, p_1 = (\text{SubWffsOf}\, p_1)_{\mathbf{2}}$.

Let us consider S, m. One can verify that $(S\text{-formulasOfMaxDepth}\, m) \setminus \text{AllFormulasOf}\, S$ is empty.

Let us consider S. Observe that $\text{AtomicFormulasOf}\, S \setminus \text{AllFormulasOf}\, S$ is empty.

We now state two propositions:

(22) $\text{Depth}(\langle l \rangle \frown p_3) > \text{Depth}\, p_3$ and $\text{Depth}(\langle \text{TheNorSymbOf}\, S \rangle \frown p_3 \frown p_4) > \text{Depth}\, p_3$ and $\text{Depth}(\langle \text{TheNorSymbOf}\, S \rangle \frown p_3 \frown p_4) > \text{Depth}\, p_4$.

(23) If p_1 is not 0-w.f.f., then $p_1 = \langle x \rangle \mathbin{^\frown} p_4 \mathbin{^\frown} p_2$ iff $x = S$-firstChar(p_1) and $p_4 = \mathrm{head}\, p_1$ and $p_2 = \mathrm{tail}\, p_1$.

Let us consider S, m_2. Observe that there exists a non 0-w.f.f. m_2-w.f.f. string of S which is non exal.

Let us consider S and let p_1 be an exal w.f.f. string of S. One can verify that tail p_1 is empty.

Let us consider S and let p_1 be a non exal non 0-w.f.f. w.f.f. string of S. Then tail p_1 is a w.f.f. string of S.

Let us consider S and let p_1 be a non exal non 0-w.f.f. w.f.f. string of S. One can check that tail p_1 is w.f.f..

Let us consider S and let p_1 be a non 0-w.f.f. non exal w.f.f. string of S. One can verify that S-firstChar$(p_1) \dot{-}$ TheNorSymbOf S is empty.

Let us consider m, S and let p_1 be an $m+1$-w.f.f. string of S. Note that head p_1 is m-w.f.f..

Let us consider m, S and let p_1 be an $m+1$-w.f.f. non exal non 0-w.f.f. string of S. Observe that tail p_1 is m-w.f.f..

One can prove the following proposition

(24) For every element I of U-InterpretersOf S holds (I, m)-TruthEval $\in$ $Boolean^{S\text{-formulasOfMaxDepth}\, m}$.

Let us consider S. One can check that there exists an of-atomic-formula element of S which is non literal.

One can prove the following proposition

(25) If $l_2 \notin \mathrm{rng}\, p$, then $((l_2, u)$ ReassignIn $I)$-TermEval$(p) = I$-TermEval(p).

Let us consider X, S, s. We say that s is X-occurring if and only if:

(Def. 38) $s \in$ SymbolsOf$(((\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\}) \cap X)$.

Let us consider S, s and let us consider X. We say that X is s-containing if and only if:

(Def. 39) $s \in$ SymbolsOf$((\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\} \cap X)$.

Let us consider X, S, s. We introduce s is X-absent as an antonym of s is X-occurring.

Let us consider S, s, X. We introduce X is s-free as an antonym of X is s-containing.

Let X be a finite set and let us consider S. Observe that there exists a literal element of S which is X-absent.

Let us consider S, t. Note that rng $t \cap$ LettersOf S is non empty.

Let us consider S, p_1. One can verify that rng $p_1 \cap$ LettersOf S is non empty.

Let us consider B, S and let A be a subset of B. Note that every element of S which is A-occurring is also B-occurring.

Let us consider A, B, S. Observe that every element of S which is A null B-absent is also $A \cap B$-absent.

Let F be a finite set and let us consider A, S. Note that every F-absent element of S which is A-absent is also $A \cup F$-absent.

Let us consider S, U and let I be an (S, U)-interpreter-like function. One can check that OwnSymbolsOf $S \setminus \mathrm{dom}\, I$ is empty.

One can prove the following proposition

(26) There exists u such that $u = I(l)(\emptyset)$ and (l, u) ReassignIn $I = I$.

Let us consider S, X. We say that X is S-covering if and only if:

(Def. 40) For every p_1 holds $p_1 \in X$ or xnot $p_1 \in X$.

Let us consider S. One can check that every set which is S-mincover is also S-covering.

Let us consider U, let p_1 be a non 0-w.f.f. non exal w.f.f. string of S, and let I be an element of U-InterpretersOf S.

One can verify that (I-TruthEval p_1)$\dot{-}$((I-TruthEval head p_1) $'$nor$'$ (I-TruthEval tail p_1)) is empty.

The functor ExFormulasOf S yielding a subset of (AllSymbolsOf S)* $\setminus \{\emptyset\}$ is defined by:

(Def. 41) ExFormulasOf $S = \{p_1; p_1$ ranges over strings of S: p_1 is w.f.f. $\wedge$ p_1 is exal$\}$.

Let us consider S. Note that ExFormulasOf S is non empty.

Let us consider S. One can check that every element of ExFormulasOf S is exal and w.f.f..

Let us consider S. Note that every element of ExFormulasOf S is w.f.f..

Let us consider S. Observe that every element of ExFormulasOf S is exal.

Let us consider S. Observe that ExFormulasOf $S \setminus$ AllFormulasOf S is empty.

Let us consider U, S_1 and let S_2 be an S_1-extending language. Note that every function which is (S_2, U)-interpreter-like is also (S_1, U)-interpreter-like.

Let us consider U, S_1, let S_2 be an S_1-extending language, and let I be an (S_2, U)-interpreter-like function. Observe that $I \restriction$ OwnSymbolsOf S_1 is (S_1, U)-interpreter-like.

Let us consider U, S_1, let S_2 be an S_1-extending language, let I_1 be an element of U-InterpretersOf S_1, and let I_2 be an (S_2, U)-interpreter-like function. Note that $I_2 {+\cdot} I_1$ is (S_2, U)-interpreter-like.

Let us consider U, S, let I be an element of U-InterpretersOf S, and let us consider X. We say that X is I-satisfied if and only if:

(Def. 42) For every p_1 such that $p_1 \in X$ holds I-TruthEval $p_1 = 1$.

Let us consider S, U, X and let I be an element of U-InterpretersOf S. We say that I satisfies X if and only if:

(Def. 43) X is I-satisfied.

Let us consider U, S, let e be an empty set, and let I be an element of U-InterpretersOf S. Observe that e null I is I-satisfied.

Let us consider X, U, S and let I be an element of U-InterpretersOf S. Observe that there exists a subset of X which is I-satisfied.

Let us consider U, S and let I be an element of U-InterpretersOf S. One can check that there exists a set which is I-satisfied.

Let us consider U, S, let I be an element of U-InterpretersOf S, and let X be an I-satisfied set. One can check that every subset of X is I-satisfied.

Let us consider U, S, let I be an element of U-InterpretersOf S, and let X, Y be I-satisfied sets. One can verify that $X \cup Y$ is I-satisfied.

Let us consider U, S, let I be an element of U-InterpretersOf S, and let X be an I-satisfied set. Observe that I null X satisfies X .

Let us consider S, X. We say that X is S-correct if and only if the condition (Def. 44) is satisfied.

(Def. 44) Let U be a non empty set, I be an element of U-InterpretersOf S, x be an I-satisfied set, and given p_1. If $\langle x, p_1 \rangle \in X$, then I-TruthEval $p_1 = 1$.

Let us consider S. Note that $\emptyset$ null S is S-correct.

Let us consider S, X. Observe that there exists a subset of X which is S-correct.

Next we state two propositions:

(27) For every element I of U-InterpretersOf S holds I-TruthEval $p_1 = 1$ iff $\{p_1\}$ is I-satisfied.

(28) s is $\{w\}$-occurring iff $s \in \operatorname{rng} w$.

Let us consider U, S, let us consider p_3, p_4, and let I be an element of U-InterpretersOf S. Observe that (I-TruthEval$\langle$TheNorSymbOf $S\rangle \frown p_3 \frown p_4) \dot{-}$ ((I-TruthEval p_3) 'nor' (I-TruthEval p_4)) is empty.

Let us consider S, p_1, U and let I be an element of U-InterpretersOf S. Note that (I-TruthEval xnot $p_1) \dot{-} \neg$(I-TruthEval p_1) is empty.

Let us consider X, S, p_1. We say that p_1 is X-implied if and only if:

(Def. 45) For every non empty set U and for every element I of U-InterpretersOf S such that X is I-satisfied holds I-TruthEval $p_1 = 1$.

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