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Preliminaries to Classical First Order Model Theory[1]

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Summary. First of a series of articles laying down the bases for classical first order model theory. These articles introduce a framework for treating arbitrary languages with equality. This framework is kept as generic and modular as possible: both the language and the derivation rule are introduced as a type, rather than a fixed functor; definitions and results regarding syntax, semantics, interpretations and sequent derivation rules, respectively, are confined to separate articles, to mark out the hierarchy of dependences among different definitions and constructions.

As an application limited to countable languages, satisfiability theorem and a full version of the Gödel completeness theorem are delivered, with respect to a fixed, remarkably thrifty, set of correct rules. Besides the self-referential significance for the Mizar project itself of those theorems being formalized with respect to a generic, equality-furnished, countable language, this is the first step to work out other milestones of model theory, such as Lowenheim-Skolem and compactness theorems. Being the receptacle of all results of broader scope stemmed during the various formalizations, this first article stays at a very generic level, with results and registrations about objects already in the Mizar Mathematical Library.

Without introducing the Language structure yet, three fundamental definitions of wide applicability are also given: the 'unambiguous' attribute (see [20], definition on page 5), the functor '-multiCat', which is the iteration of '^' over a FinSequence of FinSequence, and the functor SubstWith, which realizes the substitution of a single symbol inside a generic FinSequence.

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The papers [11], [2], [4], [12], [23], [7], [13], [19], [22], [14], [15], [10], [16], [9], [25], [1], [27], [8], [24], [6], [3], [5], [17], [28], [30], [29], [21], [26], and [18] provide the notation and terminology for this paper.

For simplicity, we adopt the following rules: U, D are non empty sets, X is a non empty subset of D, d is an element of D, A, B, C, Y, x, y, z are sets, f is a binary operation on D, i, m, n are natural numbers, and g is a function.

Let X be a set and let f be a function. We say that f is X-one-to-one if and only if:

(Def. 1) $f{\restriction}X$ is one-to-one.

Let us consider D, f and let X be a set. We say that X is f-unambiguous if and only if:

(Def. 2) f is $X \times D$-one-to-one.

Let us consider D and let X be a set. We say that X is D-prefix if and only if:

(Def. 3) X is (the concatenation of D)-unambiguous.

Let D be a set. The functor D-pr1 yielding a binary operation on D is defined by:

(Def. 4) $D\text{-pr1} = \pi_1(D \times D)$.

One can prove the following propositions:

(1) $A^m \cap B^* = A^m \cap B^m$.

(2) $A^m \cap B^* = (A \cap B)^m$.

(3) $(A \cap B)^m = A^m \cap B^m$.

(4) For all finite sequences x, y such that x is U-valued and y is U-valued holds (the concatenation of U)$(x, y) = x \frown y$.

(5) For every set x holds x is a non empty finite sequence of elements of D iff $x \in D^* \setminus \{\emptyset\}$.

Let D be a non empty set. One can check that D-pr1 is associative.

Let D be a set. Note that there exists a binary operation on D which is associative.

Let X be a set and let Y be a subset of X. Then Y^* is a non empty subset of X^*.

Let D be a non empty set. Observe that the concatenation of D is associative. Observe that $D^* \setminus \{\emptyset\}$ is non empty.

Let m be a natural number. Note that there exists an element of D^* which is m-element.

Let X be a set and let f be a function. Let us observe that f is X-one-to-one if and only if:

(Def. 5) For all sets x, y such that $x, y \in X \cap \operatorname{dom} f$ and $f(x) = f(y)$ holds $x = y$.

Let us consider D, f. Note that there exists a set which is f-unambiguous.

Let f be a function and let x be a set. Note that $f{\restriction}\{x\}$ is one-to-one.

One can verify that every set which is empty is also empty-membered. Let e be an empty set. Note that $\{e\}$ is empty-membered.

Let us consider U and let m_1 be a non zero natural number. Observe that U^{m_1} has non empty elements.

Let X be an empty-membered set. Note that every subset of X is empty-membered.

Let us consider A and let m_0 be a zero number. Note that A^{m_0} is empty-membered.

Let e be an empty set and let m_1 be a non zero natural number. Observe that e^{m_1} is empty.

Let us consider D, f and let e be an empty set. One can verify that $e \cap f$ is f-unambiguous.

Let us consider U and let e be an empty set. One can check that $e \cap U$ is U-prefix.

Let us consider U. Observe that there exists a set which is U-prefix.

Let us consider D, f and let x be a finite sequence of elements of D. The functor $\mathrm{MultPlace}(f, x)$ yields a function and is defined by:

(Def. 6) $\operatorname{dom} \mathrm{MultPlace}(f, x) = \mathbb{N}$ and $(\mathrm{MultPlace}(f, x))(0) = x(1)$ and for every natural number n holds $(\mathrm{MultPlace}(f, x))(n + 1) = f((\mathrm{MultPlace}(f, x))(n), x(n + 2))$.

Let us consider D, f and let x be an element of $D^* \setminus \{\emptyset\}$. The functor $\mathrm{MultPlace}(f, x)$ yields a function and is defined as follows:

(Def. 7) $\mathrm{MultPlace}(f, x) = \mathrm{MultPlace}(f, (x$ **qua** element of $D^*))$.

Let us consider D, f. The functor $\mathrm{MultPlace}\, f$ yielding a function from $D^* \setminus \{\emptyset\}$ into D is defined as follows:

(Def. 8) For every element x of $D^* \setminus \{\emptyset\}$ holds $(\mathrm{MultPlace}\, f)(x) = (\mathrm{MultPlace}(f, x))(\operatorname{len} x - 1)$.

Let us consider D, f and let X be a set. Let us observe that X is f-unambiguous if and only if:

(Def. 9) For all sets x, y, d_1, d_2 such that $x, y \in X \cap D$ and $d_1, d_2 \in D$ and $f(x, d_1) = f(y, d_2)$ holds $x = y$ and $d_1 = d_2$.

Let us consider D. The functor D-firstChar yields a function from $D^* \setminus \{\emptyset\}$ into D and is defined as follows:

(Def. 10) $D\text{-firstChar} = \mathrm{MultPlace}(D\text{-pr1})$.

One can prove the following proposition

(6) For every finite sequence p such that p is U-valued and non empty holds $U\text{-firstChar}(p) = p(1)$.

Let us consider D. The functor D-multiCat yielding a function is defined as follows:

(Def. 11) D-multiCat $= (\emptyset \dot{\longmapsto} \emptyset) +\cdot$ MultPlace (the concatenation of D).

Let us consider D. Then D-multiCat is a function from $(D^*)^*$ into D^*.

Let us consider D and let e be an empty set. One can check that D-multiCat(e) is empty.

Let us consider D. Observe that every subset of D^1 is D-prefix.

The following propositions are true:

(7) If A is D-prefix, then D-multiCat$^\circ A^m$ is D-prefix.

(8) If A is D-prefix, then D-multiCat is A^m-one-to-one.

(9) $Y^{m+1} \subseteq Y^* \setminus \{\emptyset\}$.

(10) If m is zero, then $Y^m = \{\emptyset\}$.

(11) $Y^i = Y^{\mathrm{Seg}\, i}$.

(12) If $x \in A^m$, then x is a finite sequence of elements of A.

Let A, X be sets. Then $\chi_{A,X}$ is a function from X into *Boolean*.

Next we state three propositions:

(13) $(\mathrm{MultPlace}\, f)(\langle d \rangle) = d$ and for every non empty finite sequence x of elements of D holds $(\mathrm{MultPlace}\, f)(x \mathbin{\frown} \langle d \rangle) = f((\mathrm{MultPlace}\, f)(x), d)$.

(14) For every non empty element d of $(D^*)^*$ holds D-multiCat$(d) =$ (MultPlace (the concatenation of D))(d).

(15) For all elements d_1, d_2 of D^* holds D-multiCat$(\langle d_1, d_2 \rangle) = d_1 \mathbin{\frown} d_2$.

Let f, g be finite sequences. One can verify that $\langle f, g \rangle$ is finite sequence-like.

Let us consider m and let f, g be m-element finite sequences. Note that $\langle f, g \rangle$ is m-element.

Let X, Y be sets, let f be an X-defined function, and let g be a Y-defined function. Observe that $\langle f, g \rangle$ is $X \cap Y$-defined.

Let X be a set and let f, g be X-defined functions. Observe that $\langle f, g \rangle$ is X-defined.

Let X, Y be sets, let f be a total X-defined function, and let g be a total Y-defined function. Note that $\langle f, g \rangle$ is total.

Let X be a set and let f, g be total X-defined functions. Note that $\langle f, g \rangle$ is total.

Let X, Y be sets, let f be an X-valued function, and let g be a Y-valued function. One can verify that $\langle f, g \rangle$ is $X \times Y$-valued.

Let us consider D. Observe that there exists a finite sequence which is D-valued.

Let us consider D, m. Note that there exists a D-valued finite sequence which is m-element.

Let X, Y be non empty sets, let f be a function from X into Y, and let p be an X-valued finite sequence. Observe that $f \cdot p$ is finite sequence-like.

Let us consider m, let f be a function from X into Y, and let p be an m-element X-valued finite sequence. Note that $f \cdot p$ is m-element.

Let us consider D, f and let p, q be elements of D^*.

The functor f AppliedPairwiseTo(p, q) yields a finite sequence of elements of D and is defined by:

(Def. 12) f AppliedPairwiseTo$(p, q) = f \cdot \langle p, q \rangle$.

Let us consider D, f, m and let p, q be m-element elements of D^*. Note that f AppliedPairwiseTo(p, q) is m-element.

Let us consider D, f and let p, q be elements of D^*. We introduce $f_\backslash(p, q)$ as a synonym of f AppliedPairwiseTo(p, q).

$\mathbb{Z}$ can be characterized by the condition:

(Def. 13) $\mathbb{Z} = \mathbb{N} \cup (\{0\} \times \mathbb{N} \setminus \{\langle 0, 0 \rangle\})$.

We now state the proposition

(16) For every finite sequence p such that p is Y-valued and m-element holds $p \in Y^m$.

Let us consider A, B. The functor $A^{\leftarrow} \cap B$ yields a subset of A and is defined by:

(Def. 14) $A^{\leftarrow} \cap B = A \cap B$.

The functor $A \cap^{\rightarrow} B$ yielding a subset of B is defined as follows:

(Def. 15) $A \cap^{\rightarrow} B = A \cap B$.

Let us consider B, A. The functor $A \operatorname{null} B$ is defined by:

(Def. 16) $A \operatorname{null} B = A$.

Let us consider A, B, C. One can check that $(B \setminus A) \cap (A \cap C)$ is empty.

Let us consider A, B. The functor $A \setminus^{\leftarrow} B$ yields a subset of A and is defined as follows:

(Def. 17) $A \setminus^{\leftarrow} B = A \setminus B$.

Let us consider A, B. The functor $A \cup^{\leftrightarrow} B$ yielding a subset of $A \cup B$ is defined by:

(Def. 18) $A \cup^{\leftrightarrow} B = A$.

For simplicity, we adopt the following convention: X is a set, P, Q, R are binary relations, f is a function, p, q are finite sequences, and U_1, U_2 are non empty sets.

Let R be a binary relation. Note that R^* is transitive and R^* is reflexive.

The function plus from $\mathbb{C}$ into $\mathbb{C}$ is defined as follows:

(Def. 19) For every complex number z holds $\operatorname{plus}(z) = z + 1$.

The following two propositions are true:

(17) If $\operatorname{rng} f \subseteq \operatorname{dom} f$, then $f^* = \bigcup\{f^{m_2} : m_2$ ranges over elements of $\mathbb{N}\}$.

(18) If $f \subseteq g$, then $f^m \subseteq g^m$.

Let X be a functional set. Note that $\bigcup X$ is relation-like.

Next we state the proposition

(19) If $Y \subseteq B^A$, then $\bigcup Y \subseteq A \times B$.

Let us consider Y. Observe that $Y \setminus Y$ is empty.

Let us consider D, d. One can check that $\{\mathrm{id}_D(d)\} \setminus \{d\}$ is empty.

One can prove the following propositions:

(20) $f = \{\langle x, f(x)\rangle; x$ ranges over elements of $\mathrm{dom}\, f : x \in \mathrm{dom}\, f\}$.

(21) For every total Y-defined binary relation R holds $\mathrm{id}_Y \subseteq R \cdot R^\smile$.

(22) $D^{m+n} =$ (the concatenation of $D)^\circ(D^m \times D^n)$.

(23) For all binary relations P, Q holds $(P \cup Q)^{-1}(Y) = P^{-1}(Y) \cup Q^{-1}(Y)$.

(24) $(\chi_{A,B})^{-1}(\{0\}) = B \setminus A$ and $(\chi_{A,B})^{-1}(\{1\}) = A \cap B$.

(25) For every non empty set y holds $y = f(x)$ iff $x \in f^{-1}(\{y\})$.

(26) If f is Y-valued and finite sequence-like, then f is a finite sequence of elements of Y.

Let us consider Y and let X be a subset of Y. Observe that every binary relation which is X-valued is also Y-valued.

Let us consider A, U. One can verify that every relation between A and U which is quasi total is also total.

The following propositions are true:

(27) Let Q be a quasi total relation between B and U_1, R be a quasi total relation between B and U_2, and P be a relation between A and B. If $P \cdot Q \cdot Q^\smile \cdot R$ is function-like, then $P \cdot Q \cdot Q^\smile \cdot R = P \cdot R$.

(28) For all finite sequences p, q such that p is non empty holds $(p \frown q)(1) = p(1)$.

Let us consider U and let p, q be U-valued finite sequences. One can check that $p \frown q$ is U-valued.

Let X be a set. We see that the finite sequence of elements of X is an element of X^*.

Let us consider U, X. Let us observe that X is U-prefix if and only if:

(Def. 20) For all U-valued finite sequences p_1, q_1, p_2, q_2 such that $p_1, p_2 \in X$ and $p_1 \frown q_1 = p_2 \frown q_2$ holds $p_1 = p_2$ and $q_1 = q_2$.

Let X be a set. Observe that every element of X^* is X-valued.

Let us consider U, m and let X be a U-prefix set. Observe that U-multiCat$^\circ\, X^m$ is U-prefix.

Next we state the proposition

(29) $X \dot{-} Y = \emptyset$ iff $X = Y$.

Let us consider x. Note that $\mathrm{id}_{\{x\}} \dot{-} \{\langle x, x\rangle\}$ is empty.

Let us consider x, y. Observe that $(x \dot{\longmapsto} y) \dot{-} \{\langle x, y\rangle\}$ is empty.

Let us consider x. Note that $\mathrm{id}_{\{x\}} \dot{-} (x \dot{\longmapsto} x)$ is empty.

The following proposition is true

(30) $x \in D^* \setminus \{\emptyset\}$ iff x is a D-valued finite sequence and non empty.

In the sequel f denotes a binary operation on D.

The following proposition is true

(31) $(\mathrm{MultPlace}\, f)(\langle d \rangle) = d$ and for every D-valued finite sequence x such that x is non empty holds $(\mathrm{MultPlace}\, f)(x \frown \langle d \rangle) = f((\mathrm{MultPlace}\, f)(x), d)$.

For simplicity, we adopt the following rules: A, B, C, X, Y, Z, x, x_1, y, y_1, y_2 are sets, U, U_1, U_2, U_3 are non empty sets, u, u_1, u_2 are elements of U, P, R are binary relations, f, g are functions, k, m, n are natural numbers, k_1, m_2, n_1 are elements of $\mathbb{N}$, m_1, n_2 are non zero natural numbers, p, p_1, p_2 are finite sequences, and q, q_1, q_2 are U-valued finite sequences.

Let us consider p, x, y. Note that $p \widetilde{+} (x, y)$ is finite sequence-like.

Let us consider x, y, p. The functor (x, y)-SymbolSubstIn p yielding a finite sequence is defined by:

(Def. 21) $(x, y)\text{-SymbolSubstIn}\, p = p \widetilde{+} (x, y)$.

Let us consider x, y, m and let p be an m-element finite sequence. Observe that (x, y)-SymbolSubstIn p is m-element.

Let us consider X. Observe that there exists a finite sequence which is X-valued.

Let us consider x, U, u and let p be a U-valued finite sequence. Observe that (x, u)-SymbolSubstIn p is U-valued.

Let us consider X, x, y and let p be an X-valued finite sequence. Then (x, y)-SymbolSubstIn p can be characterized by the condition:

(Def. 22) $(x, y)\text{-SymbolSubstIn}\, p = (\mathrm{id}_X +\cdot (x, y)) \cdot p$.

Let us consider U, x, u, q. Then (x, u)-SymbolSubstIn q is a finite sequence of elements of U.

Let us consider U, x, u. The functor x SubstWith u yielding a function from U^* into U^* is defined as follows:

(Def. 23) For every q holds $(x\, \mathrm{SubstWith}\, u)(q) = (x, u)\text{-SymbolSubstIn}\, q$.

Let us consider U, x, u. Note that x SubstWith u is finite sequence-yielding.

Let F be a finite sequence-yielding function and let x be a set. Observe that $F(x)$ is finite sequence-like.

Let us consider U, x, u, m and let p be a U-valued m-element finite sequence. Note that $(x\, \mathrm{SubstWith}\, u)(p)$ is m-element.

Let e be an empty set. One can verify that $(x\, \mathrm{SubstWith}\, u)(e)$ is empty.

Let us consider U. Note that U-multiCat is finite sequence-yielding.

One can verify that there exists a U-valued finite sequence which is non empty.

Let us consider U, m_1, n and let p be an $m_1 + n$-element U-valued finite sequence. Observe that $\{p(m_1)\} \setminus U$ is empty.

Let us consider U, m, n and let p be an $m + 1 + n$-element element of U^*. One can check that $\{p(m + 1)\} \setminus U$ is empty.

Let us consider x. Note that $\langle x \rangle \dot{-} \{\langle 1, x \rangle\}$ is empty.

Let us consider m and let p be an $m+1$-element finite sequence. Observe that $(p \restriction \mathrm{Seg}\, m) \frown \langle p(m+1) \rangle \dot{-} p$ is empty.

Let us consider m, n and let p be an $m+n$-element finite sequence. One can verify that $p \restriction \mathrm{Seg}\, m$ is m-element.

Let us observe that every binary relation which is $\{\emptyset\}$-valued is also empty yielding and every binary relation which is empty yielding is also $\{\emptyset\}$-valued.

The following two propositions are true:

(32) U-multiCat$(x) = (\mathrm{MultPlace}\,(\text{the concatenation of } U))(x)$.

(33) If p is U^*-valued, then U-multiCat$(p \frown \langle q \rangle) = U$-multiCat$(p) \frown q$.

Let us consider Y, let X be a subset of Y, and let R be a total Y-defined binary relation. One can check that $R \restriction X$ is total.

The following propositions are true:

(34) If $u = u_1$, then (u_1, x_2)-SymbolSubstIn$\langle u \rangle = \langle x_2 \rangle$ and if $u \neq u_1$, then (u_1, x_2)-SymbolSubstIn$\langle u \rangle = \langle u \rangle$.

(35) If $u = u_1$, then $(u_1\, \mathrm{SubstWith}\, u_2)(\langle u \rangle) = \langle u_2 \rangle$ and if $u \neq u_1$, then $(u_1\, \mathrm{SubstWith}\, u_2)(\langle u \rangle) = \langle u \rangle$.

(36) $(x\, \mathrm{SubstWith}\, u)(q_1 \frown q_2) = (x\, \mathrm{SubstWith}\, u)(q_1) \frown (x\, \mathrm{SubstWith}\, u)(q_2)$.

(37) If p is U^*-valued,
then $(x\, \mathrm{SubstWith}\, u)(U\text{-multiCat}(p)) = U\text{-multiCat}((x\, \mathrm{SubstWith}\, u) \cdot p)$.

(38) (The concatenation of $U)^{\circ}(\mathrm{id}_{U^1}) = \{\langle u, u \rangle : u$ ranges over elements of $U\}$.

Let us consider f, U, u. One can verify that $(f \restriction U)(u) \dot{-} f(u)$ is empty.

Let us consider f, U_1, U_2, let u be an element of U_1, and let g be a function from U_1 into U_2. Observe that $(f \cdot g)(u) \dot{-} f(g(u))$ is empty.

One can verify that every integer number which is non negative is also natural.

Let x, y be real numbers. One can verify that $\max(x, y) - x$ is non negative.

The following proposition is true

(39) If x is boolean, then $x = 1$ iff $x \neq 0$.

Let us consider Y and let X be a subset of Y. Note that $X \setminus Y$ is empty.

Let us consider x, y. Observe that $\{x\} \setminus \{x, y\}$ is empty and $\langle x, y \rangle_{\mathbf{1}} \dot{-} x$ is empty.

Let us consider x, y. Observe that $\langle x, y \rangle_{\mathbf{2}} \dot{-} y$ is empty.

Let n be a positive natural number and let X be a non empty set. Note that there exists an element of $X^* \setminus \{\emptyset\}$ which is n-element.

Let us consider m_1. One can verify that every finite sequence which is m_1+0-element is also non empty.

Let us consider R, x. Note that $R\, \mathrm{null}\, x$ is relation-like.

Let f be a function-like set and let us consider x. One can check that $f\, \mathrm{null}\, x$ is function-like.

Let p be a finite sequence-like binary relation and let us consider x. One can check that $p\,\mathrm{null}\,x$ is finite sequence-like.

Let us consider p, x. Observe that $p\,\mathrm{null}\,x$ is $\mathrm{len}\,p$-element.

Let p be a non empty finite sequence. Note that $\mathrm{len}\,p$ is non zero.

Let R be a binary relation and let X be a set. Observe that $R{\restriction}X$ is X-defined.

Let us consider x and let e be an empty set. Observe that $e\,\mathrm{null}\,x$ is empty.

Let us consider X and let e be an empty set. One can verify that $e\,\mathrm{null}\,X$ is X-valued.

Let Y be a non empty finite sequence-membered set. One can check that every function which is Y-valued is also finite sequence-yielding.

Let us consider X, Y. Note that every element of $(Y^*)^X$ is finite sequence-yielding.

We now state the proposition

(40) If f is X^*-valued, then $f(x) \in X^*$.

Let us consider m, n and let p be an m-element finite sequence. Observe that $p\,\mathrm{null}\,n$ is $\mathrm{Seg}\,m+n$-defined.

Let us consider m, n, let p be an m-element finite sequence, and let q be an n-element finite sequence. Observe that $p \frown q$ is $m+n$-element.

The following two propositions are true:

(41) Let p_1, p_2, q_1, q_2 be finite sequences. Suppose p_1 is m-element but q_1 is m-element but $p_1 \frown p_2 = q_1 \frown q_2$ or $p_2 \frown p_1 = q_2 \frown q_1$. Then $p_1 = q_1$ and $p_2 = q_2$.

(42) If U-multiCat(x) is U_1-valued and $x \in (U^*)^*$, then x is a finite sequence of elements of ${U_1}^*$.

Let us consider U. One can verify that there exists a reflexive binary relation on U which is total.

Let us consider m. Note that every finite sequence which is $m+1$-element is also non empty.

Let us consider U, u. Note that $\mathrm{id}_U(u) \dot{-} u$ is empty.

Let us consider U and let p be a U-valued non empty finite sequence. Observe that $\{p(1)\} \setminus U$ is empty.

Next we state the proposition

(43) If $x_1 = x_2$, then $f{+\cdot}(x_1 \dot{\longmapsto} y_1){+\cdot}(x_2 \dot{\longmapsto} y_2) = f{+\cdot}(x_2 \dot{\longmapsto} y_2)$ and if $x_1 \neq x_2$, then $f{+\cdot}(x_1 \dot{\longmapsto} y_1){+\cdot}(x_2 \dot{\longmapsto} y_2) = f{+\cdot}(x_2 \dot{\longmapsto} y_2){+\cdot}(x_1 \dot{\longmapsto} y_1)$.

Let us consider X, U. Note that there exists an X-defined function which is U-valued and total.

Let us consider X, U, let P be a U-valued total X-defined binary relation, and let Q be a total U-defined binary relation. One can verify that $P \cdot Q$ is total.

We now state the proposition

(44) If $p \frown p_1 \frown p_2$ is X-valued, then p_2 is X-valued and p_1 is X-valued and p is X-valued.

Let us consider X and let R be a binary relation. One can check that $R \operatorname{null} X$ is $X \cup \operatorname{rng} R$-valued.

Let X, Y be functional sets. One can verify that $X \cup Y$ is functional.

Let us note that every set which is finite sequence-membered is also finite-membered.

Let X be a functional set. The functor SymbolsOf X is defined by:

(Def. 24) $\quad$ SymbolsOf $X = \bigcup\{\operatorname{rng} x; x$ ranges over elements of $X \cup \{\emptyset\} : x \in X\}$.

Let us observe that there exists a set which is trivial, finite sequence-membered, and non empty.

Let X be a functional finite finite-membered set. Note that SymbolsOf X is finite.

Let X be a finite finite sequence-membered set. One can verify that SymbolsOf X is finite.

The following proposition is true

(45) $\quad$ SymbolsOf$\{f\} = \operatorname{rng} f$.

Let z be a non zero complex number. One can check that $|z|$ is positive.

The scheme *Sc1* deals with a set $\mathcal{A}$, a set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding a set, and states that:

$\{\mathcal{F}(x); x$ ranges over elements of $\mathcal{A} : x \in \mathcal{A}\} = \{\mathcal{F}(x); x$ ranges over elements of $\mathcal{B} : x \in \mathcal{A}\}$

provided the following condition is satisfied:

- $\mathcal{A} \subseteq \mathcal{B}$.

Let X be a functional set. Then SymbolsOf X can be characterized by the condition:

(Def. 25) $\quad$ SymbolsOf $X = \bigcup\{\operatorname{rng} x; x$ ranges over elements of X: $x \in X\}$.

One can prove the following propositions:

(46) $\quad$ For every functional set B and for every subset A of B holds SymbolsOf $A \subseteq$ SymbolsOf B.

(47) $\quad$ For all functional sets A, B holds SymbolsOf$(A \cup B) =$ SymbolsOf $A \cup$ SymbolsOf B.

Let us consider X and let F be a subset of 2^X. One can verify that $\bigcup F \setminus X$ is empty.

The following four propositions are true:

(48) $\quad X = (X \setminus Y) \cup X \cap Y$.

(49) $\quad$ If A^m meets B^n, then $m = n$.

(50) $\quad$ If B is D-prefix and $A \subseteq B$, then A is D-prefix.

(51) $\quad f \subseteq g$ iff for every x such that $x \in \operatorname{dom} f$ holds $x \in \operatorname{dom} g$ and $f(x) = g(x)$.

Let us consider U. One can verify that every element of $(U^* \setminus \{\emptyset\})^*$ which is non empty is also non empty yielding.

Let e be an empty set. One can verify that every element of e^* is empty.

The following proposition is true

(52)(i) If U_1-multiCat$(x) \neq \emptyset$ and U_2-multiCat$(x) \neq \emptyset$, then U_1-multiCat$(x) = U_2$-multiCat(x),

(ii) if p is $\emptyset^*$-valued, then U_1-multiCat$(p) = \emptyset$, and

(iii) if U_1-multiCat$(p) = \emptyset$ and p is ${U_1}^*$-valued, then p is $\emptyset^*$-valued.

Let us consider U, x. Note that U-multiCat(x) is U-valued.

Let us consider x. The functor $x\,\mathrm{null}$ is defined by:

(Def. 26) $x\,\mathrm{null} = x$.

Let Y be a set with non empty elements. Observe that every Y-valued binary relation which is non empty is also non empty yielding.

Let us consider X. Observe that $X \setminus \{\emptyset\}$ has non empty elements.

Let X be a set with non empty elements. One can check that every subset of X has non empty elements.

Let us consider U. Note that U^* is infinite. Observe that U^* has a non-empty element.

Let X be a set with a non-empty element. Note that there exists a subset of X which is non empty and has non empty elements.

One can prove the following propositions:

(53) If $U_1 \subseteq U_2$ and $Y \subseteq {U_1}^*$ and p is Y-valued and $p \neq \emptyset$ and Y has non empty elements, then U_1-multiCat$(p) = U_2$-multiCat(p).

(54) If there exists p such that $x = p$ and p is X^*-valued, then U-multiCat(x) is X-valued.

Let us consider X, m. Observe that $X^m \setminus X^*$ is empty.

The following two propositions are true:

(55) $(A \cap B)^* = A^* \cap B^*$.

(56) $(P \cup Q){\restriction}X = P{\restriction}X \cup Q{\restriction}X$.

Let us consider X. One can check that $2^X \setminus X$ is non empty.

Let us consider X and let R be a binary relation. One can verify that $R\,\mathrm{null}\,X$ is $X \cup \mathrm{dom}\,R$-defined.

Next we state the proposition

(57) $f{\restriction}X {+\cdot} g = f{\restriction}(X \setminus \mathrm{dom}\,g) \cup g$.

We now state the proposition

(58) If $y \notin \pi_2(X)$, then $A \times \{y\}$ misses X.

Let us consider X. The functor X-freeCountableSet is defined by:

(Def. 27) X-freeCountableSet $= \mathbb{N} \times \{$the element of $2^{\pi_2(X)} \setminus \pi_2(X)\}$.

Next we state the proposition

(59) X-freeCountableSet $\cap X = \emptyset$ and X-freeCountableSet is infinite.

Let us consider X. Observe that X-freeCountableSet is infinite. Observe that X-freeCountableSet $\cap X$ is empty. One can verify that X-freeCountableSet is countable.

One can check that $\mathbb{N} \setminus \mathbb{Z}$ is empty.

Let us consider x, p. Observe that $(\langle x \rangle \frown p)(1) \dot{-} x$ is empty.

Let us consider m, let m_0 be a zero number, and let p be an m-element finite sequence. Note that $p \operatorname{null} m_0$ is total.

Let us consider U, q_1, q_2. One can check that U-multiCat$(\langle q_1, q_2 \rangle) \dot{-} q_1 \frown q_2$ is empty.

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