

Brouwer Fixed Point Theorem in the General Case

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Summary. In this article we prove the Brouwer fixed point theorem for an arbitrary convex compact subset of \mathcal{E}^n with a non empty interior. This article is based on [15].

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The notation and terminology used here have been introduced in the following papers: [17], [12], [1], [4], [7], [16], [6], [13], [10], [2], [3], [14], [9], [20], [18], [8], [19], [11], [21], and [5].

1. PRELIMINARIES

For simplicity, we adopt the following convention: n is a natural number, p, q, u, w are points of \mathcal{E}_T^n , S is a subset of \mathcal{E}_T^n , A, B are convex subsets of \mathcal{E}_T^n , and r is a real number.

Next we state several propositions:

- (1) $(1 - r) \cdot p + r \cdot q = p + r \cdot (q - p)$.
- (2) If $u, w \in \text{halfline}(p, q)$ and $|u - p| = |w - p|$, then $u = w$.
- (3) Let given S . Suppose $p \in S$ and $p \neq q$ and $S \cap \text{halfline}(p, q)$ is Bounded. Then there exists w such that
 - (i) $w \in \text{Fr } S \cap \text{halfline}(p, q)$,
 - (ii) for every u such that $u \in S \cap \text{halfline}(p, q)$ holds $|p - u| \leq |p - w|$, and
 - (iii) for every r such that $r > 0$ there exists u such that $u \in S \cap \text{halfline}(p, q)$ and $|w - u| < r$.

- (4) For every A such that A is closed and $p \in \text{Int } A$ and $p \neq q$ and $A \cap \text{halfline}(p, q)$ is Bounded there exists u such that $\text{Fr } A \cap \text{halfline}(p, q) = \{u\}$.
- (5) If $r > 0$, then $\text{Fr } \overline{\text{Ball}}(p, r) = \text{Sphere}(p, r)$.

Let n be an element of \mathbb{N} , let A be a Bounded subset of $\mathcal{E}_{\mathbb{T}}^n$, and let p be a point of $\mathcal{E}_{\mathbb{T}}^n$. One can verify that $p + A$ is Bounded.

2. MAIN THEOREMS

Next we state four propositions:

- (6) Let n be an element of \mathbb{N} and A be a convex subset of $\mathcal{E}_{\mathbb{T}}^n$. Suppose A is compact and non boundary. Then there exists a function h from $\mathcal{E}_{\mathbb{T}}^n \upharpoonright A$ into $\text{Tdisk}(0_{\mathcal{E}_{\mathbb{T}}^n}, 1)$ such that h is homeomorphism and $h^\circ \text{Fr } A = \text{Sphere}((0_{\mathcal{E}_{\mathbb{T}}^n}), 1)$.
- (7) Let given A, B . Suppose A is compact and non boundary and B is compact and non boundary. Then there exists a function h from $\mathcal{E}_{\mathbb{T}}^n \upharpoonright A$ into $\mathcal{E}_{\mathbb{T}}^n \upharpoonright B$ such that h is homeomorphism and $h^\circ \text{Fr } A = \text{Fr } B$.
- (8)¹ For every A such that A is compact and non boundary holds every continuous function from $\mathcal{E}_{\mathbb{T}}^n \upharpoonright A$ into $\mathcal{E}_{\mathbb{T}}^n \upharpoonright A$ has a fixpoint.
- (9) Let A be a non empty convex subset of $\mathcal{E}_{\mathbb{T}}^n$. Suppose A is compact and non boundary. Let F_1 be a non empty subspace of $\mathcal{E}_{\mathbb{T}}^n \upharpoonright A$. If $\Omega_{(F_1)} = \text{Fr } A$, then F_1 is not a retract of $\mathcal{E}_{\mathbb{T}}^n \upharpoonright A$.

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