

# Brouwer Fixed Point Theorem for Simplexes

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**Summary.** In this article we prove the Brouwer fixed point theorem for an arbitrary simplex which is the convex hull of its  $n + 1$  affinely independent vertices of  $\mathcal{E}^n$ . First we introduce the Lebesgue number, which for an arbitrary open cover of a compact metric space  $\mathfrak{M}$  is a positive real number so that any ball of about such radius must be completely contained in a member of the cover. Then we introduce the notion of a bounded simplicial complex and the diameter of a bounded simplicial complex. We also prove the estimation of diameter decrease which is connected with the barycentric subdivision. Finally, we prove the Brouwer fixed point theorem and compute the small inductive dimension of  $\mathcal{E}^n$ . This article is based on [16].

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The papers [7], [31], [1], [8], [11], [17], [30], [14], [20], [4], [13], [9], [32], [21], [5], [19], [2], [3], [6], [22], [24], [18], [35], [26], [29], [33], [23], [27], [28], [34], [15], [25], [12], and [10] provide the terminology and notation for this paper.

## 1. THE LEBESGUE NUMBER

In this paper  $M$  is a non empty metric space and  $F, G$  are open families of subsets of  $M_{\text{top}}$ .

Let us consider  $M$ . Let us assume that  $M_{\text{top}}$  is compact. Let  $F$  be a family of subsets of  $M_{\text{top}}$ . Let us assume that  $F$  is open and  $F$  is a cover of  $M_{\text{top}}$ . A positive real number is said to be a Lebesgue number of  $F$  if:

(Def. 1) For every point  $p$  of  $M$  there exists a subset  $A$  of  $M_{\text{top}}$  such that  $A \in F$  and  $\text{Ball}(p, \text{it}) \subseteq A$ .

In the sequel  $L$  denotes a Lebesgue number of  $F$ .

Next we state three propositions:

- (1) If  $M_{\text{top}}$  is compact and  $F$  is a cover of  $M_{\text{top}}$  and  $F \subseteq G$ , then  $L$  is a Lebesgue number of  $G$ .
- (2) If  $M_{\text{top}}$  is compact and  $F$  is a cover of  $M_{\text{top}}$  and finer than  $G$ , then  $L$  is a Lebesgue number of  $G$ .
- (3) Let  $L_1$  be a positive real number. Suppose  $M_{\text{top}}$  is compact and  $F$  is a cover of  $M_{\text{top}}$  and  $L_1 \leq L$ . Then  $L_1$  is a Lebesgue number of  $F$ .

## 2. BOUNDED SIMPLICIAL COMPLEXES

In the sequel  $n, k$  denote natural numbers,  $X$  denotes a set, and  $K$  denotes a simplicial complex structure.

Let us consider  $M$ . One can check that every subset of  $M$  which is finite is also bounded.

Next we state the proposition

- (4) For every finite non empty subset  $S$  of  $M$  there exist points  $p, q$  of  $M$  such that  $p, q \in S$  and  $\rho(p, q) = \emptyset S$ .

Let us consider  $M, K$ . We say that  $K$  is  $M$ -bounded if and only if:

- (Def. 2) There exists  $r$  such that for every  $A$  such that  $A \in$  the topology of  $K$  holds  $A$  is bounded and  $\emptyset A \leq r$ .

The following proposition is true

- (5) Let  $K$  be a non void simplicial complex structure. If  $K$  is  $M$ -bounded and  $A$  is a simplex of  $K$ , then  $A$  is bounded.

Let us consider  $M, X$ . Note that there exists a simplicial complex of  $X$  which is  $M$ -bounded and non void.

Let us consider  $M$ . Note that there exists a simplicial complex structure which is  $M$ -bounded, non void, subset-closed, and finite-membered.

Let us consider  $M, X$  and let  $K$  be an  $M$ -bounded simplicial complex str of  $X$ . Note that every sub simplicial complex of  $K$  is  $M$ -bounded.

Let us consider  $M, X$ , let  $K$  be an  $M$ -bounded subset-closed simplicial complex str of  $X$ , and let  $i$  be an integer. One can verify that the skeleton of  $K$  and  $i$  is  $M$ -bounded.

The following proposition is true

- (6) If  $K$  is finite-vertices, then  $K$  is  $M$ -bounded.

3. THE DIAMETER OF A BOUNDED SIMPLICIAL COMPLEX

Let us consider  $M$  and let  $K$  be a simplicial complex structure. Let us assume that  $K$  is  $M$ -bounded. The functor  $\text{diameter}(M, K)$  yielding a real number is defined by:

- (Def. 3)(i) For every  $A$  such that  $A \in$  the topology of  $K$  holds  $\varnothing A \leq \text{diameter}(M, K)$  and for every  $r$  such that for every  $A$  such that  $A \in$  the topology of  $K$  holds  $\varnothing A \leq r$  holds  $r \geq \text{diameter}(M, K)$  if the topology of  $K$  meets  $2^{\Omega_M}$ ,
- (ii)  $\text{diameter}(M, K) = 0$ , otherwise.

One can prove the following three propositions:

- (7) If  $K$  is  $M$ -bounded, then  $0 \leq \text{diameter}(M, K)$ .
- (8) For every  $M$ -bounded simplicial complex  $\text{str } K$  of  $X$  and for every sub simplicial complex  $K_1$  of  $K$  holds  $\text{diameter}(M, K_1) \leq \text{diameter}(M, K)$ .
- (9) Let  $K$  be an  $M$ -bounded subset-closed simplicial complex  $\text{str}$  of  $X$  and  $i$  be an integer. Then  $\text{diameter}(M, \text{the skeleton of } K \text{ and } i) \leq \text{diameter}(M, K)$ .

Let us consider  $M$  and let  $K$  be an  $M$ -bounded non void subset-closed simplicial complex structure. Then  $\text{diameter}(M, K)$  is a real number and it can be characterized by the condition:

- (Def. 4)(i) For every  $A$  such that  $A$  is a simplex of  $K$  holds  $\varnothing A \leq \text{diameter}(M, K)$ , and
- (ii) for every  $r$  such that for every  $A$  such that  $A$  is a simplex of  $K$  holds  $\varnothing A \leq r$  holds  $r \geq \text{diameter}(M, K)$ .

Next we state the proposition

- (10) For every finite subset  $S$  of  $M$  holds  $\text{diameter}(M, \text{the complex of } \{S\}) = \varnothing S$ .

Let us consider  $n$  and let  $K$  be a simplicial complex  $\text{str}$  of  $\mathcal{E}_T^n$ . We say that  $K$  is bounded if and only if:

- (Def. 5)  $K$  is  $\mathcal{E}^n$ -bounded.

The functor  $\varnothing K$  yielding a real number is defined as follows:

- (Def. 6)  $\varnothing K = \text{diameter}(\mathcal{E}^n, K)$ .

Let us consider  $n$ . One can verify the following observations:

- \* every simplicial complex  $\text{str}$  of  $\mathcal{E}_T^n$  which is bounded is also  $\mathcal{E}^n$ -bounded,
- \* there exists a simplicial complex of  $\mathcal{E}_T^n$  which is bounded, affinely independent, simplex-join-closed, non void, finite-degree, and total, and
- \* every simplicial complex  $\text{str}$  of  $\mathcal{E}_T^n$  which is finite-vertices is also bounded.

## 4. THE ESTIMATION OF DIAMETER OF THE BARYCENTRIC SUBDIVISION

In the sequel  $V$  is a real linear space.

The following two propositions are true:

- (11) Let  $S$  be a simplex of BCS  $K_2$  and  $F$  be a  $\subseteq$ -linear finite finite-membered family of subsets of  $V$ . Suppose  $S = (\text{the center of mass } V)^\circ F$  and  $\bigcup F$  is a simplex of  $K_2$ . Let  $a_1, a_2$  be vectors of  $V$ . Suppose  $a_1, a_2 \in S$ . Then there exist vectors  $b_1, b_2$  of  $V$  and there exists a real number  $r$  such that  $b_1 \in \text{Vertices BCS}(\text{the complex of } \{\bigcup F\})$  and  $b_2 \in \text{Vertices BCS}(\text{the complex of } \{\bigcup F\})$  and  $a_1 - a_2 = r \cdot (b_1 - b_2)$  and  $0 \leq r \leq \frac{\overline{\bigcup_{F-1}}}{\overline{\bigcup F}}$ .
- (12) Let  $A$  be an affinely independent subset of  $\mathcal{E}_T^n$  and  $E$  be an enumeration of  $A$ . If  $\text{dom } E \setminus X$  is non empty, then  $\text{conv } E^\circ X = \bigcap \{\text{conv } A \setminus \{E(k)\}; k \text{ ranges over elements of } \mathbb{N}: k \in \text{dom } E \setminus X\}$ .

In the sequel  $A$  denotes a subset of  $\mathcal{E}_T^n$ .

The following three propositions are true:

- (13) For every bounded subset  $a$  of  $\mathcal{E}^n$  such that  $a = A$  and for every point  $p$  of  $\mathcal{E}^n$  such that  $p \in \text{conv } A$  holds  $\text{conv } A \subseteq \overline{\text{Ball}}(p, \emptyset a)$ .
- (14)  $A$  is Bounded iff  $\text{conv } A$  is Bounded.
- (15) For all bounded subsets  $a, c_1$  of  $\mathcal{E}^n$  such that  $c_1 = \text{conv } A$  and  $a = A$  holds  $\emptyset a = \emptyset c_1$ .

Let us consider  $n$  and let  $K$  be a bounded simplicial complex str of  $\mathcal{E}_T^n$ . Observe that every subdivision str of  $K$  is bounded.

The following propositions are true:

- (16) For every bounded finite-degree non void simplicial complex  $K$  of  $\mathcal{E}_T^n$  such that  $|K| \subseteq \Omega_K$  holds  $\emptyset \text{ BCS } K \leq \frac{\text{degree}(K)}{\text{degree}(K)+1} \cdot \emptyset K$ .
- (17) For every bounded finite-degree non void simplicial complex  $K$  of  $\mathcal{E}_T^n$  such that  $|K| \subseteq \Omega_K$  holds  $\emptyset \text{ BCS}(k, K) \leq \left(\frac{\text{degree}(K)}{\text{degree}(K)+1}\right)^k \cdot \emptyset K$ .
- (18) Let  $K$  be a bounded finite-degree non void simplicial complex of  $\mathcal{E}_T^n$ . If  $|K| \subseteq \Omega_K$ , then for every  $r$  such that  $r > 0$  there exists  $k$  such that  $\emptyset \text{ BCS}(k, K) < r$ .
- (19) Let  $i, j$  be elements of  $\mathbb{N}$ . Then there exists a function  $f$  from  $\mathcal{E}_T^i \times \mathcal{E}_T^j$  into  $\mathcal{E}_T^{i+j}$  such that  $f$  is homeomorphism and for every element  $f_1$  of  $\mathcal{E}_T^i$  and for every element  $f_2$  of  $\mathcal{E}_T^j$  holds  $f(f_1, f_2) = f_1 \wedge f_2$ .
- (20) Let  $i, j$  be elements of  $\mathbb{N}$  and  $f$  be a function from  $\mathcal{E}_T^i \times \mathcal{E}_T^j$  into  $\mathcal{E}_T^{i+j}$ . Suppose that for every element  $f_1$  of  $\mathcal{E}_T^i$  and for every element  $f_2$  of  $\mathcal{E}_T^j$  holds  $f(f_1, f_2) = f_1 \wedge f_2$ . Let given  $r, f_1$  be a point of  $\mathcal{E}^i, f_2$  be a point of  $\mathcal{E}^j$ , and  $f_3$  be a point of  $\mathcal{E}^{i+j}$ . If  $f_3 = f_1 \wedge f_2$ , then  $f^\circ(\text{OpenHypercube}(f_1, r) \times \text{OpenHypercube}(f_2, r)) = \text{OpenHypercube}(f_3, r)$ .

- (21)  $A$  is Bounded iff there exists a point  $p$  of  $\mathcal{E}^n$  and there exists  $r$  such that  $A \subseteq \text{OpenHypercube}(p, r)$ .

Let us consider  $n$ . Observe that every subset of  $\mathcal{E}_T^n$  which is closed and Bounded is also compact.

Let us consider  $n$  and let  $A$  be an affinely independent subset of  $\mathcal{E}_T^n$ . One can verify that  $\text{conv } A$  is compact.

## 5. MAIN THEOREMS

Next we state the proposition

- (22) Let  $A$  be a non empty affinely independent subset of  $\mathcal{E}_T^n$ ,  $E$  be an enumeration of  $A$ , and  $F$  be a finite sequence of elements of  $2^{\text{the carrier of } \mathcal{E}_T^n \upharpoonright \text{conv } A}$ . Suppose  $\text{len } F = \overline{\overline{A}}$  and  $\text{rng } F$  is closed and for every subset  $S$  of  $\text{dom } F$  holds  $\text{conv } E^\circ S \subseteq \bigcup (F^\circ S)$ . Then  $\bigcap \text{rng } F$  is non empty.

In the sequel  $A$  denotes an affinely independent subset of  $\mathcal{E}_T^n$ .

Next we state four propositions:

- (23) Let given  $A$ . Suppose  $\overline{\overline{A}} = n + 1$ . Let  $f$  be a continuous function from  $\mathcal{E}_T^n \upharpoonright \text{conv } A$  into  $\mathcal{E}_T^n \upharpoonright \text{conv } A$ . Then there exists a point  $p$  of  $\mathcal{E}_T^n$  such that  $p \in \text{dom } f$  and  $f(p) = p$ .
- (24) For every  $A$  such that  $\overline{\overline{A}} = n + 1$  holds every continuous function from  $\mathcal{E}_T^n \upharpoonright \text{conv } A$  into  $\mathcal{E}_T^n \upharpoonright \text{conv } A$  has a fixpoint.
- (25) If  $\overline{\overline{A}} = n + 1$ , then  $\text{ind conv } A = n$ .
- (26)  $\text{ind}(\mathcal{E}_T^n) = n$ .

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