

# Set of Points on Elliptic Curve in Projective Coordinates<sup>1</sup>

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**Summary.** In this article, we formalize a set of points on an elliptic curve over  $\mathbf{GF}(\mathbf{p})$ . Elliptic curve cryptography [10], whose security is based on a difficulty of discrete logarithm problem of elliptic curves, is important for information security.

MML identifier: EC\_PF\_1, version: 7.11.07 4.160.1126

The notation and terminology used here have been introduced in the following papers: [15], [1], [16], [13], [3], [8], [5], [6], [19], [18], [14], [17], [2], [12], [4], [9], [22], [23], [20], [21], [11], and [7].

## 1. FINITE PRIME FIELD $\mathbf{GF}(\mathbf{p})$

For simplicity, we use the following convention:  $x$  is a set,  $i, j$  are integers,  $n, n_1, n_2$  are natural numbers, and  $K, K_1, K_2$  are fields.

Let  $K$  be a field. A field is called a subfield of  $K$  if it satisfies the conditions (Def. 1).

- (Def. 1)(i) The carrier of it  $\subseteq$  the carrier of  $K$ ,
- (ii) the addition of it = (the addition of  $K$ )  $\upharpoonright$  (the carrier of it),
  - (iii) the multiplication of it = (the multiplication of  $K$ )  $\upharpoonright$  (the carrier of it),
  - (iv)  $1_{\text{it}} = 1_K$ , and
  - (v)  $0_{\text{it}} = 0_K$ .

We now state two propositions:

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<sup>1</sup>This work was supported by JSPS KAKENHI 22300285.

- (1)  $K$  is a subfield of  $K$ .
- (2) Let  $S_1$  be a non empty double loop structure. Suppose that
  - (i) the carrier of  $S_1$  is a subset of the carrier of  $K$ ,
  - (ii) the addition of  $S_1 = (\text{the addition of } K) \upharpoonright (\text{the carrier of } S_1)$ ,
  - (iii) the multiplication of  $S_1 = (\text{the multiplication of } K) \upharpoonright (\text{the carrier of } S_1)$ ,
  - (iv)  $1_{(S_1)} = 1_K$ ,
  - (v)  $0_{(S_1)} = 0_K$ , and
  - (vi)  $S_1$  is right complementable, commutative, almost left invertible, and non degenerated.

Then  $S_1$  is a subfield of  $K$ .

Let  $K$  be a field. One can check that there exists a subfield of  $K$  which is strict.

In the sequel  $S_2, S_3$  denote subfields of  $K$  and  $e_1, e_2$  denote elements of  $K$ .

We now state several propositions:

- (3) If  $K_1$  is a subfield of  $K_2$ , then for every  $x$  such that  $x \in K_1$  holds  $x \in K_2$ .
- (4) For all strict fields  $K_1, K_2$  such that  $K_1$  is a subfield of  $K_2$  and  $K_2$  is a subfield of  $K_1$  holds  $K_1 = K_2$ .
- (5) Let  $K_1, K_2, K_3$  be strict fields. Suppose  $K_1$  is a subfield of  $K_2$  and  $K_2$  is a subfield of  $K_3$ . Then  $K_1$  is a subfield of  $K_3$ .
- (6)  $S_2$  is a subfield of  $S_3$  iff the carrier of  $S_2 \subseteq$  the carrier of  $S_3$ .
- (7)  $S_2$  is a subfield of  $S_3$  iff for every  $x$  such that  $x \in S_2$  holds  $x \in S_3$ .
- (8) For all strict subfields  $S_2, S_3$  of  $K$  holds  $S_2 = S_3$  iff the carrier of  $S_2 =$  the carrier of  $S_3$ .
- (9) For all strict subfields  $S_2, S_3$  of  $K$  holds  $S_2 = S_3$  iff for every  $x$  holds  $x \in S_2$  iff  $x \in S_3$ .

Let  $K$  be a finite field. Observe that there exists a subfield of  $K$  which is finite. Then  $\overline{\overline{K}}$  is an element of  $\mathbb{N}$ .

Let us mention that there exists a field which is strict and finite.

Next we state the proposition

- (10) For every strict finite field  $K$  and for every strict subfield  $S_2$  of  $K$  such that  $\overline{\overline{K}} = \overline{\overline{S_2}}$  holds  $S_2 = K$ .

Let  $I_1$  be a field. We say that  $I_1$  is prime if and only if:

- (Def. 2) If  $K_1$  is a strict subfield of  $I_1$ , then  $K_1 = I_1$ .

Let  $p$  be a prime number. We introduce  $\text{GF}(p)$  as a synonym of  $\mathbb{Z}_p^{\text{R}}$ . One can check that  $\text{GF}(p)$  is finite. One can check that  $\text{GF}(p)$  is prime.

One can check that there exists a field which is prime.

2. ARITHMETIC IN  $\mathbf{GF}(p)$ 

In the sequel  $b, c$  denote elements of  $\mathbf{GF}(p)$  and  $F$  denotes a finite sequence of elements of  $\mathbf{GF}(p)$ .

Next we state a number of propositions:

- (11)  $0 = 0_{\mathbf{GF}(p)}$ .
- (12)  $1 = 1_{\mathbf{GF}(p)}$ .
- (13) There exists  $n_1$  such that  $a = n_1 \bmod p$ .
- (14) There exists  $a$  such that  $a = i \bmod p$ .
- (15) If  $a = i \bmod p$  and  $b = j \bmod p$ , then  $a + b = (i + j) \bmod p$ .
- (16) If  $a = i \bmod p$ , then  $-a = (p - i) \bmod p$ .
- (17) If  $a = i \bmod p$  and  $b = j \bmod p$ , then  $a - b = (i - j) \bmod p$ .
- (18) If  $a = i \bmod p$  and  $b = j \bmod p$ , then  $a \cdot b = i \cdot j \bmod p$ .
- (19) If  $a = i \bmod p$  and  $i \cdot j \bmod p = 1$ , then  $a^{-1} = j \bmod p$ .
- (20)  $a = 0$  or  $b = 0$  iff  $a \cdot b = 0$ .
- (21)  $a^0 = 1_{\mathbf{GF}(p)}$  and  $a^0 = 1$ .
- (22)  $a^2 = a \cdot a$ .
- (23) If  $a = n_1 \bmod p$ , then  $a^n = n_1^n \bmod p$ .
- (24)  $a^{n+1} = a^n \cdot a$ .
- (25) If  $a \neq 0$ , then  $a^n \neq 0$ .
- (26) Let  $F$  be an Abelian add-associative right zeroed right complementable associative commutative well unital almost left invertible distributive non empty double loop structure and  $x, y$  be elements of  $F$ . Then  $x \cdot x = y \cdot y$  if and only if  $x = y$  or  $x = -y$ .
- (27) For every prime number  $p$  and for every element  $x$  of  $\mathbf{GF}(p)$  such that  $2 < p$  and  $x + x = 0_{\mathbf{GF}(p)}$  holds  $x = 0_{\mathbf{GF}(p)}$ .
- (28)  $a^n \cdot b^n = (a \cdot b)^n$ .
- (29) If  $a \neq 0$ , then  $(a^{-1})^n = (a^n)^{-1}$ .
- (30)  $a^{n_1} \cdot a^{n_2} = a^{n_1+n_2}$ .
- (31)  $(a^{n_1})^{n_2} = a^{n_1 \cdot n_2}$ .

Let us consider  $p$ . One can verify that  $\text{MultGroup}(\mathbf{GF}(p))$  is cyclic.

The following two propositions are true:

- (32) Let  $x$  be an element of  $\text{MultGroup}(\mathbf{GF}(p))$ ,  $x_1$  be an element of  $\mathbf{GF}(p)$ , and  $n$  be a natural number. If  $x = x_1$ , then  $x^n = x_1^n$ .
- (33) There exists an element  $g$  of  $\mathbf{GF}(p)$  such that for every element  $a$  of  $\mathbf{GF}(p)$  if  $a \neq 0_{\mathbf{GF}(p)}$ , then there exists a natural number  $n$  such that  $a = g^n$ .

### 3. RELATION BETWEEN LEGENDRE SYMBOL AND THE NUMBER OF ROOTS IN $\mathbf{GF}(p)$

Let us consider  $p, a$ . We say that  $a$  is quadratic residue if and only if:

(Def. 3)  $a \neq 0$  and there exists an element  $x$  of  $\mathbf{GF}(p)$  such that  $x^2 = a$ .

We say that  $a$  is not quadratic residue if and only if:

(Def. 4)  $a \neq 0$  and it is not true that there exists an element  $x$  of  $\mathbf{GF}(p)$  such that  $x^2 = a$ .

One can prove the following proposition

(34) If  $a \neq 0$ , then  $a^2$  is quadratic residue.

Let  $p$  be a prime number. Observe that  $1_{\mathbf{GF}(p)}$  is quadratic residue.

Let us consider  $p, a$ . The functor  $\text{Lege}_p a$  yields an integer and is defined as follows:

(Def. 5) 
$$\text{Lege}_p a = \begin{cases} 0, & \text{if } a = 0, \\ 1, & \text{if } a \text{ is quadratic residue,} \\ -1, & \text{otherwise.} \end{cases}$$

Next we state several propositions:

(35)  $a$  is not quadratic residue iff  $\text{Lege}_p a = -1$ .

(36)  $a$  is quadratic residue iff  $\text{Lege}_p a = 1$ .

(37)  $a = 0$  iff  $\text{Lege}_p a = 0$ .

(38) If  $a \neq 0$ , then  $\text{Lege}_p(a^2) = 1$ .

(39)  $\text{Lege}_p(a \cdot b) = \text{Lege}_p a \cdot \text{Lege}_p b$ .

(40) If  $a \neq 0$  and  $n \bmod 2 = 0$ , then  $\text{Lege}_p(a^n) = 1$ .

(41) If  $n \bmod 2 = 1$ , then  $\text{Lege}_p(a^n) = \text{Lege}_p a$ .

(42) If  $2 < p$ , then  $\overline{\{b : b^2 = a\}} = 1 + \text{Lege}_p a$ .

### 4. SET OF POINTS ON AN ELLIPTIC CURVE OVER $\mathbf{GF}(p)$

Let  $K$  be a field. The functor  $\text{ProjCo } K$  yields a non empty subset of (the carrier of  $K$ )  $\times$  (the carrier of  $K$ )  $\times$  (the carrier of  $K$ ) and is defined by:

(Def. 6)  $\text{ProjCo } K = ((\text{the carrier of } K) \times (\text{the carrier of } K) \times (\text{the carrier of } K)) \setminus \{\langle 0_K, 0_K, 0_K \rangle\}$ .

One can prove the following proposition

(43)  $\text{ProjCo } \mathbf{GF}(p) = ((\text{the carrier of } \mathbf{GF}(p)) \times (\text{the carrier of } \mathbf{GF}(p)) \times (\text{the carrier of } \mathbf{GF}(p))) \setminus \{\langle 0, 0, 0 \rangle\}$ .

In the sequel  $P_1, P_2, P_3$  are elements of  $\mathbf{GF}(p)$ .

Let  $p$  be a prime number and let  $a, b$  be elements of  $\mathbf{GF}(p)$ . The functor  $\text{Disc}(a, b, p)$  yields an element of  $\mathbf{GF}(p)$  and is defined as follows:

(Def. 7) For all elements  $g_4, g_{27}$  of  $\text{GF}(p)$  such that  $g_4 = 4 \pmod p$  and  $g_{27} = 27 \pmod p$  holds  $\text{Disc}(a, b, p) = g_4 \cdot a^3 + g_{27} \cdot b^2$ .

Let  $p$  be a prime number and let  $a, b$  be elements of  $\text{GF}(p)$ . The functor  $\text{EC WEqProjCo}(a, b, p)$  yielding a function from  $(\text{the carrier of } \text{GF}(p)) \times (\text{the carrier of } \text{GF}(p)) \times (\text{the carrier of } \text{GF}(p))$  into  $\text{GF}(p)$  is defined by the condition (Def. 8).

(Def. 8) Let  $P$  be an element of  $(\text{the carrier of } \text{GF}(p)) \times (\text{the carrier of } \text{GF}(p)) \times (\text{the carrier of } \text{GF}(p))$ . Then  $(\text{EC WEqProjCo}(a, b, p))(P) = (P_2)^2 \cdot P_3 - ((P_1)^3 + a \cdot P_1 \cdot (P_3)^2 + b \cdot (P_3)^3)$ .

We now state the proposition

(44) For all elements  $X, Y, Z$  of  $\text{GF}(p)$  holds  $(\text{EC WEqProjCo}(a, b, p))(\langle X, Y, Z \rangle) = Y^2 \cdot Z - (X^3 + a \cdot X \cdot Z^2 + b \cdot Z^3)$ .

Let  $p$  be a prime number and let  $a, b$  be elements of  $\text{GF}(p)$ . The functor  $\text{EC SetProjCo}(a, b, p)$  yielding a non empty subset of  $\text{ProjCo GF}(p)$  is defined by:

(Def. 9)  $\text{EC SetProjCo}(a, b, p) = \{P \in \text{ProjCo GF}(p) : (\text{EC WEqProjCo}(a, b, p))(P) = 0_{\text{GF}(p)}\}$ .

One can prove the following two propositions:

(45)  $\langle 0, 1, 0 \rangle$  is an element of  $\text{EC SetProjCo}(a, b, p)$ .

(46) Let  $p$  be a prime number and  $a, b, X, Y$  be elements of  $\text{GF}(p)$ . Then  $Y^2 = X^3 + a \cdot X + b$  if and only if  $\langle X, Y, 1 \rangle$  is an element of  $\text{EC SetProjCo}(a, b, p)$ .

Let  $p$  be a prime number and let  $P, Q$  be elements of  $\text{ProjCo GF}(p)$ . We say that  $P \text{ EQ } Q$  if and only if:

(Def. 10) There exists an element  $a$  of  $\text{GF}(p)$  such that  $a \neq 0_{\text{GF}(p)}$  and  $P_1 = a \cdot Q_1$  and  $P_2 = a \cdot Q_2$  and  $P_3 = a \cdot Q_3$ .

Let us notice that the predicate  $P \text{ EQ } Q$  is reflexive and symmetric.

We now state two propositions:

(47) For every prime number  $p$  and for all elements  $P, Q, R$  of  $\text{ProjCo GF}(p)$  such that  $P \text{ EQ } Q$  and  $Q \text{ EQ } R$  holds  $P \text{ EQ } R$ .

(48) Let  $p$  be a prime number,  $a, b$  be elements of  $\text{GF}(p)$ ,  $P, Q$  be elements of  $(\text{the carrier of } \text{GF}(p)) \times (\text{the carrier of } \text{GF}(p)) \times (\text{the carrier of } \text{GF}(p))$ , and  $d$  be an element of  $\text{GF}(p)$ . Suppose  $p > 3$  and  $\text{Disc}(a, b, p) \neq 0_{\text{GF}(p)}$  and  $P \in \text{EC SetProjCo}(a, b, p)$  and  $d \neq 0_{\text{GF}(p)}$  and  $Q_1 = d \cdot P_1$  and  $Q_2 = d \cdot P_2$  and  $Q_3 = d \cdot P_3$ . Then  $Q \in \text{EC SetProjCo}(a, b, p)$ .

Let  $p$  be a prime number. The functor  $\mathbb{R}\text{-ProjCo } p$  yielding a binary relation on  $\text{ProjCo GF}(p)$  is defined by:

(Def. 11)  $\mathbb{R}\text{-ProjCo } p = \{\langle P, Q \rangle; P \text{ ranges over elements of } \text{ProjCo GF}(p), Q \text{ ranges over elements of } \text{ProjCo GF}(p) : P \text{ EQ } Q\}$ .

One can prove the following proposition

- (49) For every prime number  $p$  and for all elements  $P, Q$  of  $\text{ProjCo GF}(p)$  holds  $P \text{ EQ } Q$  iff  $\langle P, Q \rangle \in \mathbb{R}\text{-ProjCo } p$ .

Let  $p$  be a prime number. Note that  $\mathbb{R}\text{-ProjCo } p$  is total, symmetric, and transitive.

Let  $p$  be a prime number and let  $a, b$  be elements of  $\text{GF}(p)$ . The functor  $\mathbb{R}\text{-EllCur}(a, b, p)$  yielding an equivalence relation of  $\text{EC SetProjCo}(a, b, p)$  is defined as follows:

(Def. 12)  $\mathbb{R}\text{-EllCur}(a, b, p) = \mathbb{R}\text{-ProjCo } p \cap \nabla_{\text{EC SetProjCo}(a, b, p)}$ .

Next we state a number of propositions:

- (50) Let  $p$  be a prime number,  $a, b$  be elements of  $\text{GF}(p)$ , and  $P, Q$  be elements of  $\text{ProjCo GF}(p)$ . Suppose  $\text{Disc}(a, b, p) \neq 0_{\text{GF}(p)}$  and  $P, Q \in \text{EC SetProjCo}(a, b, p)$ . Then  $P \text{ EQ } Q$  if and only if  $\langle P, Q \rangle \in \mathbb{R}\text{-EllCur}(a, b, p)$ .
- (51) Let  $p$  be a prime number,  $a, b$  be elements of  $\text{GF}(p)$ , and  $P$  be an element of  $\text{ProjCo GF}(p)$ . Suppose  $p > 3$  and  $\text{Disc}(a, b, p) \neq 0_{\text{GF}(p)}$  and  $P \in \text{EC SetProjCo}(a, b, p)$  and  $P_3 \neq 0$ . Then there exists an element  $Q$  of  $\text{ProjCo GF}(p)$  such that  $Q \in \text{EC SetProjCo}(a, b, p)$  and  $Q \text{ EQ } P$  and  $Q_3 = 1$ .
- (52) Let  $p$  be a prime number,  $a, b$  be elements of  $\text{GF}(p)$ , and  $P$  be an element of  $\text{ProjCo GF}(p)$ . Suppose  $p > 3$  and  $\text{Disc}(a, b, p) \neq 0_{\text{GF}(p)}$  and  $P \in \text{EC SetProjCo}(a, b, p)$  and  $P_3 = 0$ . Then there exists an element  $Q$  of  $\text{ProjCo GF}(p)$  such that  $Q \in \text{EC SetProjCo}(a, b, p)$  and  $Q \text{ EQ } P$  and  $Q_1 = 0$  and  $Q_2 = 1$  and  $Q_3 = 0$ .
- (53) Let  $p$  be a prime number,  $a, b$  be elements of  $\text{GF}(p)$ , and  $x$  be a set. Suppose  $p > 3$  and  $\text{Disc}(a, b, p) \neq 0_{\text{GF}(p)}$  and  $x \in \text{Classes } \mathbb{R}\text{-EllCur}(a, b, p)$ . Then
- (i) there exists an element  $P$  of  $\text{ProjCo GF}(p)$  such that  $P \in \text{EC SetProjCo}(a, b, p)$  and  $P = \langle 0, 1, 0 \rangle$  and  $x = [P]_{\mathbb{R}\text{-EllCur}(a, b, p)}$ , or
  - (ii) there exists an element  $P$  of  $\text{ProjCo GF}(p)$  and there exist elements  $X, Y$  of  $\text{GF}(p)$  such that  $P \in \text{EC SetProjCo}(a, b, p)$  and  $P = \langle X, Y, 1 \rangle$  and  $x = [P]_{\mathbb{R}\text{-EllCur}(a, b, p)}$ .
- (54) Let  $p$  be a prime number and  $a, b$  be elements of  $\text{GF}(p)$ . Suppose  $p > 3$  and  $\text{Disc}(a, b, p) \neq 0_{\text{GF}(p)}$ . Then  $\text{Classes } \mathbb{R}\text{-EllCur}(a, b, p) = \{[\langle 0, 1, 0 \rangle]_{\mathbb{R}\text{-EllCur}(a, b, p)}\} \cup \{[P]_{\mathbb{R}\text{-EllCur}(a, b, p)}; P \text{ ranges over elements of } \text{ProjCo GF}(p) : P \in \text{EC SetProjCo}(a, b, p) \wedge \bigvee_{X, Y : \text{element of } \text{GF}(p)} P = \langle X, Y, 1 \rangle\}$ .
- (55) Let  $p$  be a prime number and  $a, b, d_1, Y_1, d_2, Y_2$  be elements of  $\text{GF}(p)$ . Suppose  $p > 3$  and  $\text{Disc}(a, b, p) \neq 0_{\text{GF}(p)}$  and  $\langle d_1, Y_1, 1 \rangle, \langle d_2, Y_2, 1 \rangle \in \text{EC SetProjCo}(a, b, p)$ . Then  $[\langle d_1, Y_1, 1 \rangle]_{\mathbb{R}\text{-EllCur}(a, b, p)} = [\langle d_2, Y_2, 1 \rangle]_{\mathbb{R}\text{-EllCur}(a, b, p)}$  if and only if  $d_1 = d_2$  and  $Y_1 = Y_2$ .

(56) Let  $p$  be a prime number,  $a, b$  be elements of  $\text{GF}(p)$ , and  $F_1, F_2$  be sets.

Suppose that

- (i)  $p > 3$ ,
- (ii)  $\text{Disc}(a, b, p) \neq 0_{\text{GF}(p)}$ ,
- (iii)  $F_1 = \{[(0, 1, 0)]_{\mathbb{R}\text{-EllCur}(a,b,p)}\}$ , and
- (iv)  $F_2 = \{[P]_{\mathbb{R}\text{-EllCur}(a,b,p)}; P \text{ ranges over elements of } \text{ProjCo GF}(p) : P \in \text{EC SetProjCo}(a, b, p) \wedge \bigvee_{X,Y:\text{element of GF}(p)} P = \langle X, Y, 1 \rangle\}$ .

Then  $F_1$  misses  $F_2$ .

(57) Let  $X$  be a non empty finite set,  $R$  be an equivalence relation of  $X$ ,  $S$  be a Classes  $R$ -valued function, and  $i$  be a set. If  $i \in \text{dom } S$ , then  $S(i)$  is a finite subset of  $X$ .

(58) Let  $X$  be a non empty set,  $R$  be an equivalence relation of  $X$ , and  $S$  be a Classes  $R$ -valued function. If  $S$  is one-to-one, then  $S$  is disjoint valued.

(59) Let  $X$  be a non empty set,  $R$  be an equivalence relation of  $X$ , and  $S$  be a Classes  $R$ -valued function. If  $S$  is onto, then  $\bigcup S = X$ .

(60) Let  $X$  be a non empty finite set,  $R$  be an equivalence relation of  $X$ ,  $S$  be a Classes  $R$ -valued function, and  $L$  be a finite sequence of elements of  $\mathbb{N}$ . Suppose  $S$  is one-to-one and onto and  $\text{dom } S = \text{dom } L$  and for every natural number  $i$  such that  $i \in \text{dom } S$  holds  $L(i) = \overline{\overline{S(i)}}$ . Then  $\overline{\overline{X}} = \sum L$ .

(61) Let  $p$  be a prime number,  $a, b, d$  be elements of  $\text{GF}(p)$ , and  $F, G$  be sets. Suppose that

- (i)  $p > 3$ ,
- (ii)  $\text{Disc}(a, b, p) \neq 0_{\text{GF}(p)}$ ,
- (iii)  $F = \{Y \in \text{GF}(p) : Y^2 = d^3 + a \cdot d + b\}$ ,
- (iv)  $F \neq \emptyset$ , and
- (v)  $G = \{[(d, Y, 1)]_{\mathbb{R}\text{-EllCur}(a,b,p)}; Y \text{ ranges over elements of } \text{GF}(p) : \langle d, Y, 1 \rangle \in \text{EC SetProjCo}(a, b, p)\}$ .

Then there exists a function from  $F$  into  $G$  which is onto and one-to-one.

(62) Let  $p$  be a prime number and  $a, b, d$  be elements of  $\text{GF}(p)$ . Suppose  $p > 3$  and  $\text{Disc}(a, b, p) \neq 0_{\text{GF}(p)}$ .

Then  $\overline{\overline{\{[(d, Y, 1)]_{\mathbb{R}\text{-EllCur}(a,b,p)}; Y \text{ ranges over elements of } \text{GF}(p)\}}}$

$$\overline{\overline{\langle d, Y, 1 \rangle \in \text{EC SetProjCo}(a, b, p)\}} = 1 + \text{Lege}_p(d^3 + a \cdot d + b).$$

(63) Let  $p$  be a prime number and  $a, b$  be elements of  $\text{GF}(p)$ . Suppose  $p > 3$  and  $\text{Disc}(a, b, p) \neq 0_{\text{GF}(p)}$ . Then there exists a finite sequence  $F$  of elements of  $\mathbb{N}$  such that

- (i)  $\text{len } F = p$ ,
- (ii) for every natural number  $n$  such that  $n \in \text{Seg } p$  there exists an element  $d$  of  $\text{GF}(p)$  such that  $d = n - 1$  and  $F(n) = 1 + \text{Lege}_p(d^3 + a \cdot d + b)$ , and
- (iii)  $\overline{\overline{\{[P]_{\mathbb{R}\text{-EllCur}(a,b,p)}; P \text{ ranges over elements of } \text{ProjCo GF}(p) : P \in \text{EC SetProjCo}(a, b, p) \wedge \bigvee_{X,Y:\text{element of GF}(p)} P = \langle X, Y, 1 \rangle\}}}$

$$\overline{\overline{\overline{\overline{\{[P]_{\mathbb{R}\text{-EllCur}(a,b,p)}; P \text{ ranges over elements of } \text{ProjCo GF}(p) : P \in \text{EC SetProjCo}(a, b, p) \wedge \bigvee_{X,Y:\text{element of GF}(p)} P = \langle X, Y, 1 \rangle\}}}}} = \sum F.$$

- (64) Let  $p$  be a prime number and  $a, b$  be elements of  $\text{GF}(p)$ . Suppose  $p > 3$  and  $\text{Disc}(a, b, p) \neq 0_{\text{GF}(p)}$ . Then there exists a finite sequence  $F$  of elements of  $\mathbb{Z}$  such that
- (i)  $\text{len } F = p$ ,
  - (ii) for every natural number  $n$  such that  $n \in \text{Seg } p$  there exists an element  $d$  of  $\text{GF}(p)$  such that  $d = n - 1$  and  $F(n) = \text{Lege}_p(d^3 + a \cdot d + b)$ , and
  - (iii)  $\overline{\text{Classes } \mathbb{R}\text{-EllCur}(a, b, p)} = 1 + p + \sum F$ .

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Received December 21, 2010