

Banach Algebra of Bounded Complex-Valued Functionals

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Summary. In this article, we describe some basic properties of the Banach algebra which is constructed from all bounded complex-valued functionals.

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The notation and terminology used in this paper are introduced in the following articles: [2], [16], [9], [14], [7], [8], [3], [18], [17], [4], [19], [5], [15], [1], [20], [12], [11], [10], [21], [13], and [6].

Let V be a complex algebra. A complex algebra is called a complex subalgebra of V if it satisfies the conditions (Def. 1).

- (Def. 1)(i) The carrier of it \subseteq the carrier of V ,
- (ii) the addition of it = (the addition of V) \upharpoonright (the carrier of it),
- (iii) the multiplication of it = (the multiplication of V) \upharpoonright (the carrier of it),
- (iv) the external multiplication of it = (the external multiplication of V) \upharpoonright ($\mathbb{C} \times$ the carrier of it),
- (v) $1_{it} = 1_V$, and
- (vi) $0_{it} = 0_V$.

We now state the proposition

- (1) Let X be a non empty set, V be a complex algebra, V_1 be a non empty subset of V , d_1, d_2 be elements of X , A be a binary operation on X , M be a function from $X \times X$ into X , and M_1 be a function from $\mathbb{C} \times X$ into X . Suppose that $V_1 = X$ and $d_1 = 0_V$ and $d_2 = 1_V$ and $A =$ (the addition of V) \upharpoonright (V_1) and $M =$ (the multiplication of V) \upharpoonright (V_1) and $M_1 =$ (the external multiplication of V) \upharpoonright ($\mathbb{C} \times V_1$) and V_1 has inverse. Then $\langle X, M, A, M_1, d_2, d_1 \rangle$ is a complex subalgebra of V .

Let V be a complex algebra. One can check that there exists a complex subalgebra of V which is strict.

Let V be a complex algebra and let V_1 be a subset of V . We say that V_1 is \mathbb{C} -additively-linearly-closed if and only if:

- (Def. 2) V_1 is add closed and has inverse and for every complex number a and for every element v of V such that $v \in V_1$ holds $a \cdot v \in V_1$.

Let V be a complex algebra and let V_1 be a subset of V . Let us assume that V_1 is \mathbb{C} -additively-linearly-closed and non empty. The functor $\text{Mult}(V_1, V)$ yielding a function from $\mathbb{C} \times V_1$ into V_1 is defined as follows:

- (Def. 3) $\text{Mult}(V_1, V) = (\text{the external multiplication of } V) \upharpoonright (\mathbb{C} \times V_1)$.

Let X be a non empty set. The functor $\mathbb{C}\text{-BoundedFunctions } X$ yielding a non empty subset of $\text{CAlgebra}(X)$ is defined by:

- (Def. 4) $\mathbb{C}\text{-BoundedFunctions } X = \{f : X \rightarrow \mathbb{C} : f \upharpoonright X \text{ is bounded}\}$.

Let X be a non empty set. Note that $\text{CAlgebra}(X)$ is scalar unital.

Let X be a non empty set. One can verify that $\mathbb{C}\text{-BoundedFunctions } X$ is \mathbb{C} -additively-linearly-closed and multiplicatively-closed.

Let V be a complex algebra. Observe that there exists a non empty subset of V which is \mathbb{C} -additively-linearly-closed and multiplicatively-closed.

Let V be a non empty CLS structure. We say that V is scalar-multiplication-cancelable if and only if:

- (Def. 5) For every complex number a and for every element v of V such that $a \cdot v = 0_V$ holds $a = 0$ or $v = 0_V$.

One can prove the following two propositions:

- (2) Let V be a complex algebra and V_1 be a \mathbb{C} -additively-linearly-closed multiplicatively-closed non empty subset of V .

Then $\langle V_1, \text{mult}(V_1, V), \text{Add}(V_1, V), \text{Mult}(V_1, V), \text{One}(V_1, V), \text{Zero}(V_1, V) \rangle$ is a complex subalgebra of V .

- (3) Let V be a complex algebra and V_1 be a complex subalgebra of V . Then

- (i) for all elements v_1, w_1 of V_1 and for all elements v, w of V such that $v_1 = v$ and $w_1 = w$ holds $v_1 + w_1 = v + w$,
- (ii) for all elements v_1, w_1 of V_1 and for all elements v, w of V such that $v_1 = v$ and $w_1 = w$ holds $v_1 \cdot w_1 = v \cdot w$,
- (iii) for every element v_1 of V_1 and for every element v of V and for every complex number a such that $v_1 = v$ holds $a \cdot v_1 = a \cdot v$,
- (iv) $\mathbf{1}_{(V_1)} = \mathbf{1}_V$, and
- (v) $0_{(V_1)} = 0_V$.

Let X be a non empty set. The \mathbb{C} -algebra of bounded functions of X yielding a complex algebra is defined by:

- (Def. 6) The \mathbb{C} -algebra of bounded functions of $X = \langle \mathbb{C}\text{-BoundedFunctions } X, \text{mult}(\mathbb{C}\text{-BoundedFunctions } X, \text{CAlgebra}(X)) \rangle$,

$\text{Add}(\mathbb{C}\text{-BoundedFunctions } X, \mathbb{C}\text{Algebra}(X)),$
 $\text{Mult}(\mathbb{C}\text{-BoundedFunctions } X, \mathbb{C}\text{Algebra}(X)),$
 $\text{One}(\mathbb{C}\text{-BoundedFunctions } X, \mathbb{C}\text{Algebra}(X)),$
 $\text{Zero}(\mathbb{C}\text{-BoundedFunctions } X, \mathbb{C}\text{Algebra}(X)).$

One can prove the following proposition

- (4) For every non empty set X holds the \mathbb{C} -algebra of bounded functions of X is a complex subalgebra of $\mathbb{C}\text{Algebra}(X)$.

Let X be a non empty set. Note that the \mathbb{C} -algebra of bounded functions of X is vector distributive and scalar unital.

Next we state several propositions:

- (5) Let X be a non empty set, F, G, H be vectors of the \mathbb{C} -algebra of bounded functions of X , and f, g, h be functions from X into \mathbb{C} . Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F + G$ if and only if for every element x of X holds $h(x) = f(x) + g(x)$.
- (6) Let X be a non empty set, a be a complex number, F, G be vectors of the \mathbb{C} -algebra of bounded functions of X , and f, g be functions from X into \mathbb{C} . Suppose $f = F$ and $g = G$. Then $G = a \cdot F$ if and only if for every element x of X holds $g(x) = a \cdot f(x)$.
- (7) Let X be a non empty set, F, G, H be vectors of the \mathbb{C} -algebra of bounded functions of X , and f, g, h be functions from X into \mathbb{C} . Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F \cdot G$ if and only if for every element x of X holds $h(x) = f(x) \cdot g(x)$.
- (8) For every non empty set X holds $0_{\text{the } \mathbb{C}\text{-algebra of bounded functions of } X} = X \mapsto 0$.
- (9) For every non empty set X holds $1_{\text{the } \mathbb{C}\text{-algebra of bounded functions of } X} = X \mapsto 1_{\mathbb{C}}$.

Let X be a non empty set and let F be a set. Let us assume that $F \in \mathbb{C}\text{-BoundedFunctions } X$. The functor $\text{modetrans}(F, X)$ yields a function from X into \mathbb{C} and is defined by:

- (Def. 7) $\text{modetrans}(F, X) = F$ and $\text{modetrans}(F, X) \upharpoonright X$ is bounded.

Let X be a non empty set and let f be a function from X into \mathbb{C} . The functor $\text{PreNorms}(f)$ yields a non empty subset of \mathbb{R} and is defined by:

- (Def. 8) $\text{PreNorms}(f) = \{|f(x)| : x \text{ ranges over elements of } X\}$.

We now state two propositions:

- (10) For every non empty set X and for every function f from X into \mathbb{C} such that $f \upharpoonright X$ is bounded holds $\text{PreNorms}(f)$ is upper bounded.
- (11) Let X be a non empty set and f be a function from X into \mathbb{C} . Then $f \upharpoonright X$ is bounded if and only if $\text{PreNorms}(f)$ is upper bounded.

Let X be a non empty set. The functor $\mathbb{C}\text{-BoundedFunctionsNorm } X$ yields a function from $\mathbb{C}\text{-BoundedFunctions } X$ into \mathbb{R} and is defined by:

(Def. 9) For every set x such that $x \in \mathbb{C}\text{-BoundedFunctions } X$ holds
 $(\mathbb{C}\text{-BoundedFunctionsNorm } X)(x) = \sup \text{PreNorms}(\text{modetrans}(x, X))$.

One can prove the following two propositions:

(13)¹ For every non empty set X and for every function f from X into \mathbb{C} such that $f \upharpoonright X$ is bounded holds $\text{modetrans}(f, X) = f$.

(14) For every non empty set X and for every function f from X into \mathbb{C} such that $f \upharpoonright X$ is bounded holds $(\mathbb{C}\text{-BoundedFunctionsNorm } X)(f) = \sup \text{PreNorms}(f)$.

Let X be a non empty set. The \mathbb{C} -normed algebra of bounded functions of X yielding a normed complex algebra structure is defined by:

(Def. 10) The \mathbb{C} -normed algebra of bounded functions of $X =$
 $\langle \mathbb{C}\text{-BoundedFunctions } X, \text{mult}(\mathbb{C}\text{-BoundedFunctions } X, \mathbb{C}\text{Algebra}(X)),$
 $\text{Add}(\mathbb{C}\text{-BoundedFunctions } X, \mathbb{C}\text{Algebra}(X)),$
 $\text{Mult}(\mathbb{C}\text{-BoundedFunctions } X, \mathbb{C}\text{Algebra}(X)),$
 $\text{One}(\mathbb{C}\text{-BoundedFunctions } X, \mathbb{C}\text{Algebra}(X)),$
 $\text{Zero}(\mathbb{C}\text{-BoundedFunctions } X, \mathbb{C}\text{Algebra}(X)), \mathbb{C}\text{-BoundedFunctionsNorm } X \rangle$.

Let X be a non empty set. One can verify that the \mathbb{C} -normed algebra of bounded functions of X is non empty.

Let X be a non empty set. One can check that the \mathbb{C} -normed algebra of bounded functions of X is unital.

We now state a number of propositions:

(15) Let W be a normed complex algebra structure and V be a complex algebra. Suppose $\langle \text{the carrier of } W, \text{the multiplication of } W, \text{the addition of } W, \text{the external multiplication of } W, \text{the one of } W, \text{the zero of } W \rangle = V$. Then W is a complex algebra.

(16) For every non empty set X holds the \mathbb{C} -normed algebra of bounded functions of X is a complex algebra.

(17) Let X be a non empty set and F be a point of the \mathbb{C} -normed algebra of bounded functions of X .

Then $(\text{Mult}(\mathbb{C}\text{-BoundedFunctions } X, \mathbb{C}\text{Algebra}(X)))(1_{\mathbb{C}}, F) = F$.

(18) For every non empty set X holds the \mathbb{C} -normed algebra of bounded functions of X is a complex linear space.

(19) For every non empty set X holds

$X \mapsto 0 = 0_{\text{the } \mathbb{C}\text{-normed algebra of bounded functions of } X}$.

(20) Let X be a non empty set, x be an element of X , f be a function from X into \mathbb{C} , and F be a point of the \mathbb{C} -normed algebra of bounded functions of X . If $f = F$ and $f \upharpoonright X$ is bounded, then $|f(x)| \leq \|F\|$.

¹The proposition (12) has been removed.

- (21) For every non empty set X and for every point F of the \mathbb{C} -normed algebra of bounded functions of X holds $0 \leq \|F\|$.
- (22) Let X be a non empty set and F be a point of the \mathbb{C} -normed algebra of bounded functions of X . Suppose $F = 0_{\text{the } \mathbb{C}\text{-normed algebra of bounded functions of } X}$. Then $0 = \|F\|$.
- (23) Let X be a non empty set, f, g, h be functions from X into \mathbb{C} , and F, G, H be points of the \mathbb{C} -normed algebra of bounded functions of X . Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F + G$ if and only if for every element x of X holds $h(x) = f(x) + g(x)$.
- (24) Let X be a non empty set, a be a complex number, f, g be functions from X into \mathbb{C} , and F, G be points of the \mathbb{C} -normed algebra of bounded functions of X . Suppose $f = F$ and $g = G$. Then $G = a \cdot F$ if and only if for every element x of X holds $g(x) = a \cdot f(x)$.
- (25) Let X be a non empty set, f, g, h be functions from X into \mathbb{C} , and F, G, H be points of the \mathbb{C} -normed algebra of bounded functions of X . Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F \cdot G$ if and only if for every element x of X holds $h(x) = f(x) \cdot g(x)$.
- (26) Let X be a non empty set, a be a complex number, and F, G be points of the \mathbb{C} -normed algebra of bounded functions of X . Then
- (i) if $\|F\| = 0$, then $F = 0_{\text{the } \mathbb{C}\text{-normed algebra of bounded functions of } X}$,
 - (ii) if $F = 0_{\text{the } \mathbb{C}\text{-normed algebra of bounded functions of } X}$, then $\|F\| = 0$,
 - (iii) $\|a \cdot F\| = |a| \cdot \|F\|$, and
 - (iv) $\|F + G\| \leq \|F\| + \|G\|$.

Let X be a non empty set. Note that the \mathbb{C} -normed algebra of bounded functions of X is right complementable, Abelian, add-associative, right zeroed, vector distributive, scalar distributive, scalar associative, scalar unital, discernible, reflexive, and complex normed space-like.

We now state two propositions:

- (27) Let X be a non empty set, f, g, h be functions from X into \mathbb{C} , and F, G, H be points of the \mathbb{C} -normed algebra of bounded functions of X . Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F - G$ if and only if for every element x of X holds $h(x) = f(x) - g(x)$.
- (28) Let X be a non empty set and s_1 be a sequence of the \mathbb{C} -normed algebra of bounded functions of X . If s_1 is CCauchy, then s_1 is convergent.

Let X be a non empty set. Observe that the \mathbb{C} -normed algebra of bounded functions of X is complete.

Next we state the proposition

- (29) For every non empty set X holds the \mathbb{C} -normed algebra of bounded functions of X is a complex Banach algebra.

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