Linear Transformations of Euclidean Topological Spaces. Part II

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Summary. We prove a number of theorems concerning various notions used in the theory of continuity of barycentric coordinates.

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The papers [2], [9], [4], [5], [6], [14], [10], [25], [13], [16], [3], [7], [12], [1], [24], [15], [21], [23], [19], [17], [8], [11], [22], [20], and [18] provide the terminology and notation for this paper.

1. Correspondence Between Euclidean Topological Space and Vector Space over $\mathbb{R}_F$

For simplicity, we follow the rules: $X$ denotes a set, $n$, $m$, $k$ denote natural numbers, $K$ denotes a field, $f$ denotes an $n$-element real-valued finite sequence, and $M$ denotes a matrix over $\mathbb{R}_F$ of dimension $n \times m$.

One can prove the following propositions:

(1) $X$ is a linear combination of the $n$-dimension vector space over $\mathbb{R}_F$ if and only if $X$ is a linear combination of $\mathcal{E}_T^n$.

(2) Let $L_2$ be a linear combination of the $n$-dimension vector space over $\mathbb{R}_F$ and $L_1$ be a linear combination of $\mathcal{E}_T^n$. If $L_1 = L_2$, then the support of $L_1$ = the support of $L_2$.

(3) Let $F$ be a finite sequence of elements of $\mathcal{E}_T^n$, $f_1$ be a function from $\mathcal{E}_T^n$ into $\mathbb{R}$, $F_1$ be a finite sequence of elements of the $n$-dimension vector space over $\mathbb{R}_F$, and $f_2$ be a function from the $n$-dimension vector space over $\mathbb{R}_F$ into $\mathbb{R}_F$. If $f_1 = f_2$ and $F = F_1$, then $f_1 F = f_2 F_1$. 
Let $F$ be a finite sequence of elements of $E^n_1$ and $F_1$ be a finite sequence of elements of the $n$-dimension vector space over $\mathbb{R}_F$. If $F_1 = F$, then $\sum F = \sum F_1$.

Let $L_2$ be a linear combination of the $n$-dimension vector space over $\mathbb{R}_F$ and $L_1$ be a linear combination of $E^n_1$. If $L_1 = L_2$, then $\sum L_1 = \sum L_2$.

Let $A_2$ be a subset of the $n$-dimension vector space over $\mathbb{R}_F$ and $A_1$ be a subset of $E^n_1$. If $A_2 = A_1$, then $\Omega_{\text{Lin}(A_1)} = \Omega_{\text{Lin}(A_2)}$.

Let $A_2$ be a subset of the $n$-dimension vector space over $\mathbb{R}_F$ and $A_1$ be a subset of $E^n_1$. Suppose $A_2 = A_1$. Then $A_2$ is linearly independent if and only if $A_1$ is linearly independent.

Let $V$ be a vector space over $K$, $W$ be a subspace of $V$, and $L$ be a linear combination of $V$. Then $L \mid$ the carrier of $W$ is a linear combination of $W$.

Let $V$ be a vector space over $K$, $A$ be a linearly independent subset of $V$, and $L_3$, $L_4$ be linear combinations of $V$. Suppose the support of $L_3 \subseteq A$ and the support of $L_4 \subseteq A$ and $\sum L_3 = \sum L_4$. Then $L_3 = L_4$.

Let $V$ be a real linear space, $W$ be a subspace of $V$, and $L$ be a linear combination of $V$. Then $L \mid$ the carrier of $W$ is a linear combination of $W$.

Let $U$ be a subspace of the $n$-dimension vector space over $\mathbb{R}_F$ and $W$ be a subspace of $E^n_1$. Suppose $\Omega_U = \Omega_W$. Then $X$ is a linear combination of $U$ if and only if $X$ is a linear combination of $W$.

Let $U$ be a subspace of the $n$-dimension vector space over $\mathbb{R}_F$, $W$ be a subspace of $E^n_1$, $L_5$ be a linear combination of $U$, and $L_6$ be a linear combination of $W$. If $L_5 = L_6$, then the support of $L_5 = \text{the support of } L_6$ and $\sum L_5 = \sum L_6$.

Let us consider $m$, $K$ and let $A$ be a subset of the $m$-dimension vector space over $K$. Note that $\text{Lin}(A)$ is finite dimensional.

**2. Correspondence Between the Mx2Tran Operator and Decomposition of a Vector in Basis**

The following propositions are true:

If $\text{rk}(M) = n$, then $M$ is an ordered basis of $\text{Lin}(\text{lines}(M))$.

Let $V$, $W$ be vector spaces over $K$, $T$ be a linear transformation from $V$ to $W$, $A$ be a subset of $V$, and $L$ be a linear combination of $A$. If $T \upharpoonright A$ is one-to-one, then $T(\sum L) = \sum(T^{\upharpoonright L})$.

Let $S$ be a subset of $\text{Seg} n$. Suppose $M \mid S$ is one-to-one and $\text{rng}(M \mid S) = \text{lines}(M)$. Then there exists a linear combination $L$ of $\text{lines}(M)$ such that $\sum L = (\text{Mx2Tran}(M))(f)$ and for every $k$ such that $k \in S$ holds $L(\text{Line}(M, k)) = \text{Seq}(f \mid M^{-1}((\{\text{Line}(M, k)\})))$. 
Suppose $M$ is without repeated line. Then there exists a linear combination $L$ of lines $(M)$ such that $\sum L = (\text{Mx2Tran } M) (f)$ and for every $k$ such that $k \in \text{dom } f$ holds $L(\text{Line}(M, k)) = f(k)$.

For every ordered basis $B$ of $\text{Lin(lines}(M))$ such that $B = M$ and for every element $M_1$ of $\text{Lin(lines}(M))$ such that $M_1 = (\text{Mx2Tran } M) (f)$ holds $M_1 \rightarrow B = f$.

$rng \text{Mx2Tran } M = \Omega_{\text{Lin(lines}(M))}$.

Let $F$ be a one-to-one finite sequence of elements of $\mathcal{E}^n_T$. Suppose $\text{rng } F$ is linearly independent. Then there exists a square matrix $M$ over $\mathbb{R}_F$ of dimension $n$ such that $M$ is invertible and $M \upharpoonright \text{len } F = F$.

Let $B$ be an ordered basis of the $n$-dimension vector space over $\mathbb{R}_F$. If $B = \text{MX2FinS}(I_{\mathbb{R}_F}^{n \times n})$, then $f \in \text{Lin(rng}(B \upharpoonright k))$ iff $f = (f \upharpoonright k) \land ((n -'k) \rightarrow 0)$.

Let $F$ be a one-to-one finite sequence of elements of $\mathcal{E}^n_T$. Suppose $\text{rng } F$ is linearly independent. Let $B$ be an ordered basis of the $n$-dimension vector space over $\mathbb{R}_F$. Suppose $B = \text{MX2FinS}(I_{\mathbb{R}_F}^{n \times n})$. Let $M$ be a square matrix over $\mathbb{R}_F$ of dimension $n$. If $M$ is invertible and $M \upharpoonright \text{len } F = F$, then $(\text{Mx2Tran } M) ^{\circ} (\Omega_{\text{Lin(rng}(B \upharpoonright \text{len } F)}) = \Omega_{\text{Lin(rng } F)}$.

Let $A, B$ be linearly independent subsets of $\mathcal{E}^n_T$. Suppose $\overline{A} = \overline{B}$. Then there exists a square matrix $M$ over $\mathbb{R}_F$ of dimension $n$ such that $M$ is invertible and $(\text{Mx2Tran } M) ^{\circ} (\Omega_{\text{Lin}(A)}) = \Omega_{\text{Lin} (B)}$.

3. Preservation of Linear and Affine Independence of Vectors by the Mx2Tran Operator

The following propositions are true:

For every linearly independent subset $A$ of $\mathcal{E}^n_T$ such that $\text{rk}(M) = n$ holds

$(\text{Mx2Tran } M) ^{\circ} A$ is linearly independent.

For every affinely independent subset $A$ of $\mathcal{E}^n_T$ such that $\text{rk}(M) = n$ holds

$(\text{Mx2Tran } M) ^{\circ} A$ is affinely independent.

Let $A$ be an affinely independent subset of $\mathcal{E}^n_T$. Suppose $\text{rk}(M) = n$. Let $v$ be an element of $\mathcal{E}^n_T$. If $v \in \text{Affin } A$, then $(\text{Mx2Tran } M) (v) \in \text{Affin}(\text{(Mx2Tran } M) ^{\circ} A)$ and for every $f$ holds $(v \rightarrow A)(f) = (\text{Mx2Tran } M) (v) \rightarrow (\text{Mx2Tran } M) ^{\circ} A)((\text{Mx2Tran } M) (f))$.

For every linearly independent subset $A$ of $\mathcal{E}^n_T$ such that $\text{rk}(M) = n$ holds

$(\text{Mx2Tran } M) ^{-1} (A)$ is linearly independent.

For every affinely independent subset $A$ of $\mathcal{E}^n_T$ such that $\text{rk}(M) = n$ holds

$(\text{Mx2Tran } M) ^{-1} (A)$ is affinely independent.
REFERENCES


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