Linear Transformations of Euclidean Topological Spaces

Karol Pąk
Institute of Informatics
University of Białystok
Poland

Summary. We introduce linear transformations of Euclidean topological spaces given by a transformation matrix. Next, we prove selected properties and basic arithmetic operations on these linear transformations. Finally, we show that a linear transformation given by an invertible matrix is a homeomorphism.

MML identifier: MATRTOP1, version: 7.11.07 4.156.1112

The papers [2], [12], [6], [26], [7], [8], [30], [21], [22], [23], [15], [31], [29], [19], [24], [3], [4], [9], [16], [5], [20], [18], [1], [14], [28], [13], [10], [25], [27], [11], and [17] provide the notation and terminology for this paper.

1. Preliminaries

For simplicity, we adopt the following rules: $X$, $Y$ denote sets, $n$, $m$, $k$, $i$ denote natural numbers, $r$ denotes a real number, $R$ denotes an element of $\mathbb{R}_F$, $K$ denotes a field, $f$, $f_1$, $f_2$, $g_1$, $g_2$ denote finite sequences, $r_1$, $r_2$, $r_3$ denote real-valued finite sequences, $c_1$, $c_2$ denote complex-valued finite sequences, and $F$ denotes a function.

Let us consider $X$, $Y$ and let $F$ be a positive yielding partial function from $X$ to $\mathbb{R}$. One can check that $F|Y$ is positive yielding.

Let us consider $X$, $Y$ and let $F$ be a negative yielding partial function from $X$ to $\mathbb{R}$. One can verify that $F|Y$ is negative yielding.

Let us consider $X$, $Y$ and let $F$ be a non-positive yielding partial function from $X$ to $\mathbb{R}$. Note that $F|Y$ is non-positive yielding.
Let us consider $X$, $Y$ and let $F$ be a non-negative yielding partial function from $X$ to $\mathbb{R}$. Note that $F|Y$ is non-negative yielding.

Let us consider $r_1$. One can check that $\sqrt{r_1}$ is finite sequence-like.

Let us consider $r_1$. The functor $\alpha_{r_1}$ yielding a finite sequence of elements of $\mathbb{R}_F$ is defined by:

(Def. 1) $\alpha_{r_1} = r_1$.

Let $p$ be a finite sequence of elements of $\mathbb{R}_F$. The functor $\alpha_p$ yields a finite sequence of elements of $\mathbb{R}$ and is defined as follows:

(Def. 2) $\alpha_p = p$.

We now state several propositions:

1. $(\alpha_{r_2} + \alpha_{r_3} = r_2 + r_3$.
2. $\sqrt{r_2 \cap r_3} = \sqrt{r_2} \cap \sqrt{r_3}$.
3. $\sqrt{r_1} = (\sqrt{r})$.
4. $\sqrt{r_1} = |r_1|$.
5. If $r_1$ is non-negative yielding, then $\sqrt{\sum r_1} \leq \sum \sqrt{r_1}$.
6. There exists $X$ such that $X \subseteq \text{dom } F$ and $\text{rng } F = \text{rng}(F|X)$ and $F|X$ is one-to-one.

Let us consider $c_1$, $X$. Observe that $c_1 - X$ is complex-valued.

Let us consider $r_1$, $X$. Observe that $r_1 - X$ is real-valued.

Let $c_1$ be a complex-valued finite subsequence. Note that $\text{Seq } c_1$ is complex-valued.

Let $r_1$ be a real-valued finite subsequence. Observe that $\text{Seq } r_1$ is real-valued.

One can prove the following propositions:

7. For every permutation $P$ of $\text{dom } f$ such that $f_1 = f : P$ there exists a permutation $Q$ of $\text{dom}(f - X)$ such that $f_1 - X = (f - X) \cdot Q$.
8. For every permutation $P$ of $\text{dom } c_1$ such that $c_2 = c_1 : P$ holds $\sum (c_2 - X) = \sum (c_1 - X)$.
9. Let $f$, $f_1$ be finite subsequences and $P$ be a permutation of $\text{dom } f$. If $f_1 = f : P$, then there exists a permutation $Q$ of $\text{dom } \text{Seq}(f_1|P^{-1}(X))$ such that $\text{Seq}(f|X) = \text{Seq}(f_1|P^{-1}(X)) \cdot Q$.
10. Let $c_1$, $c_2$ be complex-valued finite subsequences and $P$ be a permutation of $\text{dom } c_1$. If $c_2 = c_1 : P$, then $\sum \text{Seq}(c_1|X) = \sum \text{Seq}(c_2|P^{-1}(X))$.
11. Let $f$ be a finite subsequence and $n$ be an element of $\mathbb{N}$. If for every $i$ holds $i + n \in X$ iff $i \in Y$, then $\text{Shift}^n f|X = \text{Shift}^n (f|Y)$.
12. There exists a subset $Y$ of $\mathbb{N}$ such that $\text{Seq}((f_1 \cap f_2)|X) = (\text{Seq}(f_1|X)) \cap \text{Seq}(f_2|Y)$ and for every $n$ such that $n > 0$ holds $n \in Y$ iff $n + \text{len } f_1 \in X \cap \text{dom}(f_1 \cap f_2)$.
13. If $\text{len } g_1 = \text{len } f_1$ and $\text{len } g_2 \leq \text{len } f_2$, then $\text{Seq}((f_1 \cap f_2)|(g_1 \cap g_2)^{-1}(X)) = (\text{Seq}(f_1|g_1^{-1}(X))) \cap \text{Seq}(f_2|g_2^{-1}(X))$. 

(14) Let $D$ be a non-empty set and $M$ be a matrix over $D$ of dimension $n \times m$. Then $M - X$ is a matrix over $D$ of dimension $n - |M^{-1}(X)| \times m$.

(15) Let $F$ be a function from $\text{Seg} \, n$ into $\text{Seg} \, n$. $D$ be a non-empty set, $M$ be a matrix over $D$ of dimension $n \times m$, and given $i$. If $i \in \text{Seg} \, \text{width} \, M$, then $(M \cdot F)_{\square}i = M_{\square}i \cdot F$.

(16) Let $A$ be a matrix over $K$ of dimension $n \times m$. Suppose $\text{rk}(A) = n$. Then there exists a matrix $B$ over $K$ of dimension $m -'n \times m$ such that $\text{rk}(A \sim B) = m$.

(17) Let $A$ be a matrix over $K$ of dimension $n \times m$. Suppose $\text{rk}(A) = n$. Then there exists a matrix $B$ over $K$ of dimension $n \times n -'m$ such that $\text{rk}(A \sim B) = n$.

For simplicity, we adopt the following convention: $f, f_1, f_2$ denote $n$-element real-valued finite sequences, $p, p_1, p_2$ denote points of $E^n_1$, $M, M_1, M_2$ denote matrices over $R_F$ of dimension $n \times m$, and $A, B$ denote square matrices over $R_F$ of dimension $n$.

2. **Linear Transformations of Euclidean Topological Spaces Given by a Transformation Matrix**

Let us consider $n, m, M$. The functor $\text{Mx2Tran} \, M$ yielding a function from $E^n_1$ into $E^m_1$ is defined by:

(Def. 3)(i) $(\text{Mx2Tran} \, M)(f) = \text{Line}(\text{LineVec}2\text{Mx}^{(\theta)}f \cdot M, 1)$ if $n \neq 0$,

(ii) $(\text{Mx2Tran} \, M)(f) = 0_{E^n_1}$, otherwise.

Let us consider $n, m, M$ and let $x$ be a set. One can check that $(\text{Mx2Tran} \, M)(x)$ is function-like and relation-like and $(\text{Mx2Tran} \, M)(x)$ is real-valued and finite sequence-like.

Let us consider $n, m, M, f$. One can check that $(\text{Mx2Tran} \, M)(f)$ is $m$-element.

One can prove the following propositions:

(18) If $1 \leq i \leq m$ and $n \neq 0$, then $(\text{Mx2Tran} \, M)(f)(i) = (\theta) \cdot M_{\square}, i$.

(19) $\text{len} \, \text{MX2FinS}(I^n_{R_F} \times n) = n$.

(20) $\text{Let } B_1 \text{ be an ordered basis of the } n\text{-dimension vector space over } R_F \text{ and } B_2 \text{ be an ordered basis of the } m\text{-dimension vector space over } R_F$. Suppose $B_1 = \text{MX2FinS}(I^n_{R_F} \times n)$ and $B_2 = \text{MX2FinS}(I^m_{R_F} \times m)$. Let $M_1$ be a matrix over $R_F$ of dimension $\text{len} \, B_1 \times \text{len} \, B_2$. If $M_1 = M$, then $\text{Mx2Tran} \, M = \text{Mx2Tran}(M_1, B_1, B_2)$.

(21) For every permutation $P$ of $\text{Seg} \, n$ holds $(\text{Mx2Tran} \, M)(f) = (\text{Mx2Tran}(M \cdot P))(f \cdot P)$ and $f \cdot P$ is an $n$-element finite sequence of elements of $R$.

(22) $(\text{Mx2Tran} \, M)(f_1 + f_2) = (\text{Mx2Tran} \, M)(f_1) + (\text{Mx2Tran} \, M)(f_2)$. 


(23) \((\text{Mx2Tran}(M))(r \cdot f) = r \cdot (\text{Mx2Tran}(M))(f)\).

(24) \((\text{Mx2Tran}(M))(f_1 - f_2) = (\text{Mx2Tran}(M))(f_1) - (\text{Mx2Tran}(M))(f_2)\).

(25) \((\text{Mx2Tran}(M_1 + M_2))(f) = (\text{Mx2Tran}(M_1))(f) + (\text{Mx2Tran}(M_2))(f)\).

(26) \((R) \cdot (\text{Mx2Tran}(M))(f) = (\text{Mx2Tran}(R \cdot M))(f)\).

(27) \((\text{Mx2Tran}(M))(p_1 + p_2) = (\text{Mx2Tran}(M))(p_1) + (\text{Mx2Tran}(M))(p_2)\).

(28) \((\text{Mx2Tran}(M))(p_1 - p_2) = (\text{Mx2Tran}(M))(p_1) - (\text{Mx2Tran}(M))(p_2)\).

(29) \((\text{Mx2Tran}(M))(0_{\mathbb{E}_T^n}) = 0_{\mathbb{E}_T^n}\).

(30) For every subset \(A\) of \(E_T^n\) holds \((\text{Mx2Tran}(M))^{\circ}(p + A) = (\text{Mx2Tran}(M))(p) + (\text{Mx2Tran}(M))^\circ A\).

(31) For every subset \(A\) of \(E_T^n\) holds \((\text{Mx2Tran}(M))^{-1}((\text{Mx2Tran}(M))(p) + A) = p + (\text{Mx2Tran}(M))^{-1}(A)\).

(32) Let \(A\) be a matrix over \(\mathbb{R}_F\) of dimension \(n \times m\) and \(B\) be a matrix over \(\mathbb{R}_F\) of dimension \(A \times k\). If if \(n = 0\), then \(m = 0\) and if \(m = 0\), then \(k = 0\), then \(\text{Mx2Tran}(B \cdot \text{Mx2Tran}(A) = \text{Mx2Tran}(A \cdot B)\).

(33) \(\text{Mx2Tran}(I_{\mathbb{R}_F}^{n \times n}) = \text{id}_{\mathbb{E}_T^n}\).

(34) If \(\text{Mx2Tran}(M_1) = \text{Mx2Tran}(M_2)\), then \(M_1 = M_2\).

(35) Let \(A\) be a matrix over \(\mathbb{R}_F\) of dimension \(n \times m\) and \(B\) be a matrix over \(\mathbb{R}_F\) of dimension \(k \times m\). Then \((\text{Mx2Tran}(A \cap B))(f \cap (k \mapsto 0)) = (\text{Mx2Tran}(A))(f)\) and \((\text{Mx2Tran}(B \cap A))( (k \mapsto 0) \cap f) = (\text{Mx2Tran}(A))(f)\).

(36) Let \(A\) be a matrix over \(\mathbb{R}_F\) of dimension \(n \times m\), \(B\) be a matrix over \(\mathbb{R}_F\) of dimension \(k \times m\), and \(g\) be a \(k\)-element real-valued finite sequence. Then \((\text{Mx2Tran}(A \cap B))(f \cap g) = (\text{Mx2Tran}(A))(f) + (\text{Mx2Tran}(B))(g)\).

(37) Let \(A\) be a matrix over \(\mathbb{R}_F\) of dimension \(n \times k\) and \(B\) be a matrix over \(\mathbb{R}_F\) of dimension \(n \times m\) such that if \(n = 0\), then \(k + m = 0\). Then \((\text{Mx2Tran}(A \cap B))(f) = (\text{Mx2Tran}(A))(f) \cap (\text{Mx2Tran}(B))(f)\).

(38) \((\text{Mx2Tran}(I_{\mathbb{R}_F}^{m \times m}|n))(f)|n = f\).

3. Selected Properties of the \text{Mx2Tran} Operator

Next we state several propositions:

(39) \(\text{Mx2Tran}(M)\) is one-to-one iff \(\text{rk}(M) = n\).

(40) \(\text{Mx2Tran}(A)\) is one-to-one iff \(\text{Det} A \neq 0_{\mathbb{R}_F}\).

(41) \(\text{Mx2Tran}(M)\) is onto iff \(\text{rk}(M) = m\).

(42) \(\text{Mx2Tran}(A)\) is onto iff \(\text{Det} A \neq 0_{\mathbb{R}_F}\).

(43) For all \(A, B\) such that \(\text{Det} A \neq 0_{\mathbb{R}_F}\) holds \((\text{Mx2Tran}(A))^{-1} = \text{Mx2Tran}(B)\) iff \(A^{-1} = B\).

(44) There exists an \(m\)-element finite sequence \(L\) of elements of \(\mathbb{R}\) such that for every \(i\) such that \(i \in \text{dom} L\) holds \(L(i) = |\partial(M[i, i])|\) and for every \(f\) holds \(|(\text{Mx2Tran}(M))(f)| \leq \sum L \cdot |f|\).
(45) There exists a real number $L$ such that $L > 0$ and for every $f$ holds $|\text{Mx2Tran}(f)| \leq L \cdot |f|$.

(46) If $\text{rk}(M) = n$, then there exists a real number $L$ such that $L > 0$ and for every $f$ holds $|f| \leq L \cdot |\text{Mx2Tran}(f)|$.

(47) $\text{Mx2Tran} M$ is continuous.

Let us consider $n$, $K$. One can check that there exists a square matrix over $K$ of dimension $n$ which is invertible.

Let us consider $n$ and let $A$ be an invertible square matrix over $\mathbb{R}_F$ of dimension $n$. Note that $\text{Mx2Tran} A$ is homeomorphism.

References


Received October 26, 2010