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Linear Transformations of Euclidean Topological Spaces

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Summary. We introduce linear transformations of Euclidean topological spaces given by a transformation matrix. Next, we prove selected properties and basic arithmetic operations on these linear transformations. Finally, we show that a linear transformation given by an invertible matrix is a homeomorphism.

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The papers [2], [12], [6], [26], [7], [8], [30], [21], [22], [23], [15], [31], [29], [19], [24], [3], [4], [9], [16], [5], [20], [18], [1], [14], [28], [13], [10], [25], [27], [11], and [17] provide the notation and terminology for this paper.

1. Preliminaries

For simplicity, we adopt the following rules: X, Y denote sets, n, m, k, i denote natural numbers, r denotes a real number, R denotes an element of $\mathbb{R}_{\rm F}$, K denotes a field, f, f_1, f_2, g_1, g_2 denote finite sequences, r_1, r_2, r_3 denote real-valued finite sequences, c_1, c_2 denote complex-valued finite sequences, and F denotes a function.

Let us consider X, Y and let F be a positive yielding partial function from X to $\mathbb{R}$. One can check that $F{\restriction}Y$ is positive yielding.

Let us consider X, Y and let F be a negative yielding partial function from X to $\mathbb{R}$. One can verify that $F{\restriction}Y$ is negative yielding.

Let us consider X, Y and let F be a non-positive yielding partial function from X to $\mathbb{R}$. Note that $F{\restriction}Y$ is non-positive yielding.

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Let us consider X, Y and let F be a non-negative yielding partial function from X to $\mathbb{R}$. Note that $F \restriction Y$ is non-negative yielding.

Let us consider r_1. One can check that $\sqrt{r_1}$ is finite sequence-like.

Let us consider r_1. The functor ${}^{@}r_1$ yielding a finite sequence of elements of $\mathbb{R}_{\mathrm{F}}$ is defined by:

(Def. 1) ${}^{@}r_1 = r_1$.

Let p be a finite sequence of elements of $\mathbb{R}_{\mathrm{F}}$. The functor ${}^{@}p$ yields a finite sequence of elements of $\mathbb{R}$ and is defined as follows:

(Def. 2) ${}^{@}p = p$.

We now state several propositions:

(1) $({}^{@}r_2) + {}^{@}r_3 = r_2 + r_3$.

(2) $\sqrt{r_2 \frown r_3} = \sqrt{r_2} \frown \sqrt{r_3}$.

(3) $\sqrt{\langle r \rangle} = \langle \sqrt{r} \rangle$.

(4) $\sqrt{r_1{}^{\mathbf{2}}} = |r_1|$.

(5) If r_1 is non-negative yielding, then $\sqrt{\sum r_1} \leq \sum \sqrt{r_1}$.

(6) There exists X such that $X \subseteq \operatorname{dom} F$ and $\operatorname{rng} F = \operatorname{rng}(F \restriction X)$ and $F \restriction X$ is one-to-one.

Let us consider c_1, X. Observe that $c_1 - X$ is complex-valued.

Let us consider r_1, X. Observe that $r_1 - X$ is real-valued.

Let c_1 be a complex-valued finite subsequence. Note that $\operatorname{Seq} c_1$ is complex-valued.

Let r_1 be a real-valued finite subsequence. Observe that $\operatorname{Seq} r_1$ is real-valued.

One can prove the following propositions:

(7) For every permutation P of $\operatorname{dom} f$ such that $f_1 = f \cdot P$ there exists a permutation Q of $\operatorname{dom}(f - X)$ such that $f_1 - X = (f - X) \cdot Q$.

(8) For every permutation P of $\operatorname{dom} c_1$ such that $c_2 = c_1 \cdot P$ holds $\sum(c_2 - X) = \sum(c_1 - X)$.

(9) Let f, f_1 be finite subsequences and P be a permutation of $\operatorname{dom} f$. If $f_1 = f \cdot P$, then there exists a permutation Q of $\operatorname{dom} \operatorname{Seq}(f_1 \restriction P^{-1}(X))$ such that $\operatorname{Seq}(f \restriction X) = \operatorname{Seq}(f_1 \restriction P^{-1}(X)) \cdot Q$.

(10) Let c_1, c_2 be complex-valued finite subsequences and P be a permutation of $\operatorname{dom} c_1$. If $c_2 = c_1 \cdot P$, then $\sum \operatorname{Seq}(c_1 \restriction X) = \sum \operatorname{Seq}(c_2 \restriction P^{-1}(X))$.

(11) Let f be a finite subsequence and n be an element of $\mathbb{N}$. If for every i holds $i + n \in X$ iff $i \in Y$, then $\operatorname{Shift}^n f \restriction X = \operatorname{Shift}^n (f \restriction Y)$.

(12) There exists a subset Y of $\mathbb{N}$ such that $\operatorname{Seq}((f_1 \frown f_2) \restriction X) = (\operatorname{Seq}(f_1 \restriction X)) \frown \operatorname{Seq}(f_2 \restriction Y)$ and for every n such that $n > 0$ holds $n \in Y$ iff $n + \operatorname{len} f_1 \in X \cap \operatorname{dom}(f_1 \frown f_2)$.

(13) If $\operatorname{len} g_1 = \operatorname{len} f_1$ and $\operatorname{len} g_2 \leq \operatorname{len} f_2$, then $\operatorname{Seq}((f_1 \frown f_2) \restriction (g_1 \frown g_2)^{-1}(X)) = (\operatorname{Seq}(f_1 \restriction g_1{}^{-1}(X))) \frown \operatorname{Seq}(f_2 \restriction g_2{}^{-1}(X))$.

(14) Let D be a non empty set and M be a matrix over D of dimension $n \times m$. Then $M - X$ is a matrix over D of dimension $n -' \overline{\overline{M^{-1}(X)}} \times m$.

(15) Let F be a function from $\mathrm{Seg}\, n$ into $\mathrm{Seg}\, n$, D be a non empty set, M be a matrix over D of dimension $n \times m$, and given i. If $i \in \mathrm{Seg\, width}\, M$, then $(M \cdot F)_{\square,i} = M_{\square,i} \cdot F$.

(16) Let A be a matrix over K of dimension $n \times m$. Suppose $\mathrm{rk}(A) = n$. Then there exists a matrix B over K of dimension $m -' n \times m$ such that $\mathrm{rk}(A \frown B) = m$.

(17) Let A be a matrix over K of dimension $n \times m$. Suppose $\mathrm{rk}(A) = m$. Then there exists a matrix B over K of dimension $n \times n -' m$ such that $\mathrm{rk}(A \frown B) = n$.

For simplicity, we adopt the following convention: f, f_1, f_2 denote n-element real-valued finite sequences, p, p_1, p_2 denote points of $\mathcal{E}^n_{\mathrm{T}}$, M, M_1, M_2 denote matrices over $\mathbb{R}_{\mathrm{F}}$ of dimension $n \times m$, and A, B denote square matrices over $\mathbb{R}_{\mathrm{F}}$ of dimension n.

2. Linear Transformations of Euclidean Topological Spaces Given by a Transformation Matrix

Let us consider n, m, M. The functor $\mathrm{Mx2Tran}\, M$ yielding a function from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathcal{E}^m_{\mathrm{T}}$ is defined by:

(Def. 3)(i) $(\mathrm{Mx2Tran}\, M)(f) = \mathrm{Line}(\mathrm{LineVec2Mx}({}^{@}f) \cdot M, 1)$ if $n \neq 0$,

(ii) $(\mathrm{Mx2Tran}\, M)(f) = 0_{\mathcal{E}^m_{\mathrm{T}}}$, otherwise.

Let us consider n, m, M and let x be a set. One can check that $(\mathrm{Mx2Tran}\, M)(x)$ is function-like and relation-like and $(\mathrm{Mx2Tran}\, M)(x)$ is real-valued and finite sequence-like.

Let us consider n, m, M, f. One can check that $(\mathrm{Mx2Tran}\, M)(f)$ is m-element.

One can prove the following propositions:

(18) If $1 \leq i \leq m$ and $n \neq 0$, then $(\mathrm{Mx2Tran}\, M)(f)(i) = ({}^{@}f) \cdot M_{\square,i}$.

(19) $\mathrm{len\, MX2FinS}(I^{n \times n}_K) = n$.

(20) Let B_1 be an ordered basis of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$ and B_2 be an ordered basis of the m-dimension vector space over $\mathbb{R}_{\mathrm{F}}$. Suppose $B_1 = \mathrm{MX2FinS}(I^{n \times n}_{\mathbb{R}_{\mathrm{F}}})$ and $B_2 = \mathrm{MX2FinS}(I^{m \times m}_{\mathbb{R}_{\mathrm{F}}})$. Let M_1 be a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension $\mathrm{len}\, B_1 \times \mathrm{len}\, B_2$. If $M_1 = M$, then $\mathrm{Mx2Tran}\, M = \mathrm{Mx2Tran}(M_1, B_1, B_2)$.

(21) For every permutation P of $\mathrm{Seg}\, n$ holds $(\mathrm{Mx2Tran}\, M)(f) = (\mathrm{Mx2Tran}(M \cdot P))(f \cdot P)$ and $f \cdot P$ is an n-element finite sequence of elements of $\mathbb{R}$.

(22) $(\mathrm{Mx2Tran}\, M)(f_1 + f_2) = (\mathrm{Mx2Tran}\, M)(f_1) + (\mathrm{Mx2Tran}\, M)(f_2)$.

(23) $(\mathrm{Mx2Tran}\, M)(r \cdot f) = r \cdot (\mathrm{Mx2Tran}\, M)(f)$.

(24) $(\mathrm{Mx2Tran}\, M)(f_1 - f_2) = (\mathrm{Mx2Tran}\, M)(f_1) - (\mathrm{Mx2Tran}\, M)(f_2)$.

(25) $(\mathrm{Mx2Tran}(M_1 + M_2))(f) = (\mathrm{Mx2Tran}\, M_1)(f) + (\mathrm{Mx2Tran}\, M_2)(f)$.

(26) $(R) \cdot (\mathrm{Mx2Tran}\, M)(f) = (\mathrm{Mx2Tran}(R \cdot M))(f)$.

(27) $(\mathrm{Mx2Tran}\, M)(p_1 + p_2) = (\mathrm{Mx2Tran}\, M)(p_1) + (\mathrm{Mx2Tran}\, M)(p_2)$.

(28) $(\mathrm{Mx2Tran}\, M)(p_1 - p_2) = (\mathrm{Mx2Tran}\, M)(p_1) - (\mathrm{Mx2Tran}\, M)(p_2)$.

(29) $(\mathrm{Mx2Tran}\, M)(0_{\mathcal{E}^n_{\mathrm{T}}}) = 0_{\mathcal{E}^m_{\mathrm{T}}}$.

(30) For every subset A of $\mathcal{E}^n_{\mathrm{T}}$ holds $(\mathrm{Mx2Tran}\, M)^{\circ}(p + A) = (\mathrm{Mx2Tran}\, M)(p) + (\mathrm{Mx2Tran}\, M)^{\circ} A$.

(31) For every subset A of $\mathcal{E}^m_{\mathrm{T}}$ holds $(\mathrm{Mx2Tran}\, M)^{-1}((\mathrm{Mx2Tran}\, M)(p)+A) = p + (\mathrm{Mx2Tran}\, M)^{-1}(A)$.

(32) Let A be a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension $n \times m$ and B be a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension $\mathrm{width}\, A \times k$. If if $n = 0$, then $m = 0$ and if $m = 0$, then $k = 0$, then $\mathrm{Mx2Tran}\, B \cdot \mathrm{Mx2Tran}\, A = \mathrm{Mx2Tran}(A \cdot B)$.

(33) $\mathrm{Mx2Tran}(I^{n \times n}_{\mathbb{R}_{\mathrm{F}}}) = \mathrm{id}_{\mathcal{E}^n_{\mathrm{T}}}$.

(34) If $\mathrm{Mx2Tran}\, M_1 = \mathrm{Mx2Tran}\, M_2$, then $M_1 = M_2$.

(35) Let A be a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension $n \times m$ and B be a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension $k \times m$. Then $(\mathrm{Mx2Tran}(A \frown B))(f \frown (k \mapsto 0)) = (\mathrm{Mx2Tran}\, A)(f)$ and $(\mathrm{Mx2Tran}(B \frown A))((k \mapsto 0) \frown f) = (\mathrm{Mx2Tran}\, A)(f)$.

(36) Let A be a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension $n \times m$, B be a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension $k \times m$, and g be a k-element real-valued finite sequence. Then $(\mathrm{Mx2Tran}(A \frown B))(f \frown g) = (\mathrm{Mx2Tran}\, A)(f) + (\mathrm{Mx2Tran}\, B)(g)$.

(37) Let A be a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension $n \times k$ and B be a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension $n \times m$ such that if $n = 0$, then $k + m = 0$. Then $(\mathrm{Mx2Tran}(A \frown B))(f) = (\mathrm{Mx2Tran}\, A)(f) \frown (\mathrm{Mx2Tran}\, B)(f)$.

(38) $(\mathrm{Mx2Tran}(I^{m \times m}_{\mathbb{R}_{\mathrm{F}}} \restriction n))(f) \restriction n = f$.

3. Selected Properties of the Mx2Tran Operator

Next we state several propositions:

(39) $\mathrm{Mx2Tran}\, M$ is one-to-one iff $\mathrm{rk}(M) = n$.

(40) $\mathrm{Mx2Tran}\, A$ is one-to-one iff $\mathrm{Det}\, A \neq 0_{\mathbb{R}_{\mathrm{F}}}$.

(41) $\mathrm{Mx2Tran}\, M$ is onto iff $\mathrm{rk}(M) = m$.

(42) $\mathrm{Mx2Tran}\, A$ is onto iff $\mathrm{Det}\, A \neq 0_{\mathbb{R}_{\mathrm{F}}}$.

(43) For all A, B such that $\mathrm{Det}\, A \neq 0_{\mathbb{R}_{\mathrm{F}}}$ holds $(\mathrm{Mx2Tran}\, A)^{-1} = \mathrm{Mx2Tran}\, B$ iff $A^{\smile} = B$.

(44) There exists an m-element finite sequence L of elements of $\mathbb{R}$ such that for every i such that $i \in \mathrm{dom}\, L$ holds $L(i) = |^{@}(M_{\square,i})|$ and for every f holds $|(\mathrm{Mx2Tran}\, M)(f)| \leq \sum L \cdot |f|$.

(45) There exists a real number L such that $L > 0$ and for every f holds $|(\mathrm{Mx2Tran}\, M)(f)| \leq L \cdot |f|$.

(46) If $\mathrm{rk}(M) = n$, then there exists a real number L such that $L > 0$ and for every f holds $|f| \leq L \cdot |(\mathrm{Mx2Tran}\, M)(f)|$.

(47) $\mathrm{Mx2Tran}\, M$ is continuous.

Let us consider n, K. One can check that there exists a square matrix over K of dimension n which is invertible.

Let us consider n and let A be an invertible square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension n. Note that $\mathrm{Mx2Tran}\, A$ is homeomorphism.

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