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The Definition of Topological Manifolds

Marco Riccardi
Via del Pero 102
54038 Montignoso, Italy

Summary. This article introduces the definition of n-locally Euclidean topological spaces and topological manifolds [13].

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The papers [8], [1], [6], [15], [7], [18], [3], [4], [17], [2], [16], [9], [19], [20], [11], [12], [10], [14], and [5] provide the terminology and notation for this paper.

1. Preliminaries

Let x, y be sets. Observe that $\{\langle x, y\rangle\}$ is one-to-one.

In the sequel n denotes a natural number.

One can prove the following two propositions:

(1) For every non empty topological space T holds T and $T \restriction \Omega_T$ are homeomorphic.

(2) Let X be a non empty subspace of $\mathcal{E}^n_{\mathrm{T}}$ and f be a function from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f is continuous. Then there exists a function g from X into $\mathcal{E}^n_{\mathrm{T}}$ such that

(i) for every point a of X and for every point b of $\mathcal{E}^n_{\mathrm{T}}$ and for every real number r such that $a = b$ and $f(a) = r$ holds $g(b) = r \cdot b$, and

(ii) g is continuous.

Let us consider n and let S be a subset of $\mathcal{E}^n_{\mathrm{T}}$. We say that S is ball if and only if:

(Def. 1) There exists a point p of $\mathcal{E}^n_{\mathrm{T}}$ and there exists a real number r such that $S = \mathrm{Ball}(p, r)$.

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Let us consider n. Observe that there exists a subset of $\mathcal{E}^n_{\mathrm{T}}$ which is ball and every subset of $\mathcal{E}^n_{\mathrm{T}}$ which is ball is also open.

Let us consider n. One can verify that there exists a subset of $\mathcal{E}^n_{\mathrm{T}}$ which is non empty and ball.

In the sequel p denotes a point of $\mathcal{E}^n_{\mathrm{T}}$ and r denotes a real number.

The following proposition is true

(3) For every open subset S of $\mathcal{E}^n_{\mathrm{T}}$ such that $p \in S$ there exists ball subset B of $\mathcal{E}^n_{\mathrm{T}}$ such that $B \subseteq S$ and $p \in B$.

Let us consider n, p, r. The functor $\mathbb{B}_r(p)$ yields a subspace of $\mathcal{E}^n_{\mathrm{T}}$ and is defined as follows:

(Def. 2) $\mathbb{B}_r(p) = \mathcal{E}^n_{\mathrm{T}} \restriction \mathrm{Ball}(p, r)$.

Let us consider n. The functor $\mathbb{B}^n$ yields a subspace of $\mathcal{E}^n_{\mathrm{T}}$ and is defined as follows:

(Def. 3) $\mathbb{B}^n = \mathbb{B}_1(0_{\mathcal{E}^n_{\mathrm{T}}})$.

Let us consider n. One can verify that $\mathbb{B}^n$ is non empty. Let us consider p and let s be a positive real number. Observe that $\mathbb{B}_s(p)$ is non empty.

The following propositions are true:

(4) The carrier of $\mathbb{B}_r(p) = \mathrm{Ball}(p, r)$.

(5) If $n \neq 0$ and p is a point of $\mathbb{B}^n$, then $|p| < 1$.

(6) Let f be a function from $\mathbb{B}^n$ into $\mathcal{E}^n_{\mathrm{T}}$. Suppose $n \neq 0$ and for every point a of $\mathbb{B}^n$ and for every point b of $\mathcal{E}^n_{\mathrm{T}}$ such that $a = b$ holds $f(a) = \frac{1}{1-|b|\cdot|b|} \cdot b$. Then f is homeomorphism.

(7) Let r be a positive real number and f be a function from $\mathbb{B}^n$ into $\mathbb{B}_r(p)$. Suppose $n \neq 0$ and for every point a of $\mathbb{B}^n$ and for every point b of $\mathcal{E}^n_{\mathrm{T}}$ such that $a = b$ holds $f(a) = r \cdot b + p$. Then f is homeomorphism.

(8) $\mathbb{B}^n$ and $\mathcal{E}^n_{\mathrm{T}}$ are homeomorphic.

In the sequel q denotes a point of $\mathcal{E}^n_{\mathrm{T}}$.

We now state three propositions:

(9) For all positive real numbers r, s holds $\mathbb{B}_r(p)$ and $\mathbb{B}_s(q)$ are homeomorphic.

(10) For every non empty ball subset B of $\mathcal{E}^n_{\mathrm{T}}$ holds B and $\Omega_{\mathcal{E}^n_{\mathrm{T}}}$ are homeomorphic.

(11) Let M, N be non empty topological spaces, p be a point of M, U be a neighbourhood of p, and B be an open subset of N. Suppose U and B are homeomorphic. Then there exists an open subset V of M and there exists an open subset S of N such that $V \subseteq U$ and $p \in V$ and V and S are homeomorphic.

2. Manifold

In the sequel M is a non empty topological space.

Let us consider n, M. We say that M is n-locally Euclidean if and only if the condition (Def. 4) is satisfied.

(Def. 4) Let p be a point of M. Then there exists a neighbourhood U of p and there exists an open subset S of $\mathcal{E}^n_{\mathrm{T}}$ such that U and S are homeomorphic.

Let us consider n. Observe that $\mathcal{E}^n_{\mathrm{T}}$ is n-locally Euclidean.

Let us consider n. Observe that there exists a non empty topological space which is n-locally Euclidean.

We now state two propositions:

(12) M is n-locally Euclidean if and only if for every point p of M there exists a neighbourhood U of p and there exists ball subset B of $\mathcal{E}^n_{\mathrm{T}}$ such that U and B are homeomorphic.

(13) M is n-locally Euclidean if and only if for every point p of M there exists a neighbourhood U of p such that U and $\Omega_{\mathcal{E}^n_{\mathrm{T}}}$ are homeomorphic.

Let us consider n. Observe that every non empty topological space which is n-locally Euclidean is also first-countable.

Let us note that every non empty topological space which is 0-locally Euclidean is also discrete and every non empty topological space which is discrete is also 0-locally Euclidean.

Let us consider n. One can verify that $\mathcal{E}^n_{\mathrm{T}}$ is second-countable.

Let us consider n. Note that there exists a non empty topological space which is second-countable, Hausdorff, and n-locally Euclidean.

Let us consider n, M. We say that M is n-manifold if and only if:

(Def. 5) M is second-countable, Hausdorff, and n-locally Euclidean.

Let us consider M. We say that M is manifold-like if and only if:

(Def. 6) There exists n such that M is n-manifold.

Let us consider n. Observe that there exists a non empty topological space which is n-manifold.

Let us consider n. One can check the following observations:

* every non empty topological space which is n-manifold is also second-countable, Hausdorff, and n-locally Euclidean,
* every non empty topological space which is second-countable, Hausdorff, and n-locally Euclidean is also n-manifold, and
* every non empty topological space which is n-manifold is also manifold-like.

Let us note that every non empty topological space which is second-countable and discrete is also 0-manifold.

Let us consider n and let M be an n-manifold non empty topological space. One can verify that every non empty subspace of M which is open is also n-manifold.

Let us note that there exists a non empty topological space which is manifold-like.

A manifold is a manifold-like non empty topological space.

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