The Definition of Topological Manifolds

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Summary. This article introduces the definition of \( n \)-locally Euclidean topological spaces and topological manifolds [13].

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The papers [8], [1], [6], [15], [7], [18], [3], [4], [17], [2], [16], [9], [19], [20], [11], [12], [10], [14], and [5] provide the terminology and notation for this paper.

1. Preliminaries

Let \( x, y \) be sets. Observe that \( \{ x, y \} \) is one-to-one.

In the sequel \( n \) denotes a natural number.

One can prove the following two propositions:

1. For every non empty topological space \( T \) holds \( T \) and \( T|\Omega_T \) are homeomorphic.

2. Let \( X \) be a non empty subspace of \( E^n_T \) and \( f \) be a function from \( X \) into \( \mathbb{R}^1 \). Suppose \( f \) is continuous. Then there exists a function \( g \) from \( X \) into \( E^n_T \) such that
   (i) for every point \( a \) of \( X \) and for every point \( b \) of \( E^n_T \) and for every real number \( r \) such that \( a = b \) and \( f(a) = r \) holds \( g(b) = r \cdot b \), and
   (ii) \( g \) is continuous.

Let us consider \( n \) and let \( S \) be a subset of \( E^n_T \). We say that \( S \) is ball if and only if:

(Def. 1) There exists a point \( p \) of \( E^n_T \) and there exists a real number \( r \) such that \( S = \text{Ball}(p, r) \).
Let us consider $n$. Observe that there exists a subset of $\mathcal{E}_T^n$ which is ball and every subset of $\mathcal{E}_T^n$ which is ball is also open.

Let us consider $n$. One can verify that there exists a subset of $\mathcal{E}_T^n$ which is non empty and ball.

In the sequel $p$ denotes a point of $\mathcal{E}_T^n$ and $r$ denotes a real number.

The following proposition is true

(3) For every open subset $S$ of $\mathcal{E}_T^n$ such that $p \in S$ there exists ball subset $B$ of $\mathcal{E}_T^n$ such that $B \subseteq S$ and $p \in B$.

Let us consider $n$, $p$, $r$. The functor $\mathbb{B}_r(p)$ yields a subspace of $\mathcal{E}_T^n$ and is defined as follows:

(Def. 2) $\mathbb{B}_r(p) = \mathcal{E}_T^n \upharpoonright \text{Ball}(p, r)$.

Let us consider $n$. The functor $\mathbb{B}^n$ yields a subspace of $\mathcal{E}_T^n$ and is defined as follows:

(Def. 3) $\mathbb{B}^n = \mathbb{B}_1(0, \mathcal{E}_T^n)$.

Let us consider $n$. One can verify that $\mathbb{B}^n$ is non empty. Let us consider $p$ and let $s$ be a positive real number. Observe that $\mathbb{B}_s(p)$ is non empty.

The following propositions are true:

(4) The carrier of $\mathbb{B}_r(p) = \text{Ball}(p, r)$.

(5) If $n \neq 0$ and $p$ is a point of $\mathbb{B}^n$, then $|p| < 1$.

(6) Let $f$ be a function from $\mathbb{B}^n$ into $\mathcal{E}_T^n$. Suppose $n \neq 0$ and for every point $a$ of $\mathbb{B}^n$ and for every point $b$ of $\mathcal{E}_T^n$ such that $a = b$ holds $f(a) = \frac{1}{1-|b|} \cdot b$. Then $f$ is homeomorphism.

(7) Let $r$ be a positive real number and $f$ be a function from $\mathbb{B}^n$ into $\mathbb{B}_r(p)$. Suppose $n \neq 0$ and for every point $a$ of $\mathbb{B}^n$ and for every point $b$ of $\mathcal{E}_T^n$ such that $a = b$ holds $f(a) = r \cdot b + p$. Then $f$ is homeomorphism.

(8) $\mathbb{B}^n$ and $\mathcal{E}_T^n$ are homeomorphic.

In the sequel $q$ denotes a point of $\mathcal{E}_T^n$.

We now state three propositions:

(9) For all positive real numbers $r$, $s$ holds $\mathbb{B}_r(p)$ and $\mathbb{B}_s(q)$ are homeomorphic.

(10) For every non empty ball subset $B$ of $\mathcal{E}_T^n$ holds $B$ and $\Omega_{\mathcal{E}_T^n}$ are homeomorphic.

(11) Let $M$, $N$ be non empty topological spaces, $p$ be a point of $M$, $U$ be a neighbourhood of $p$, and $B$ be an open subset of $N$. Suppose $U$ and $B$ are homeomorphic. Then there exists an open subset $V$ of $M$ and there exists an open subset $S$ of $N$ such that $V \subseteq U$ and $p \in V$ and $V$ and $S$ are homeomorphic.
2. Manifold

In the sequel $M$ is a non empty topological space.
Let us consider $n$, $M$. We say that $M$ is $n$-locally Euclidean if and only if the condition (Def. 4) is satisfied.

(Def. 4) Let $p$ be a point of $M$. Then there exists a neighbourhood $U$ of $p$ and there exists an open subset $S$ of $E^n_T$ such that $U$ and $S$ are homeomorphic.

Let us consider $n$. Observe that $E^n_T$ is $n$-locally Euclidean.
Let us consider $n$. Observe that there exists a non empty topological space which is $n$-locally Euclidean.

We now state two propositions:

(12) $M$ is $n$-locally Euclidean if and only if for every point $p$ of $M$ there exists a neighbourhood $U$ of $p$ and there exists ball subset $B$ of $E^n_T$ such that $U$ and $B$ are homeomorphic.

(13) $M$ is $n$-locally Euclidean if and only if for every point $p$ of $M$ there exists a neighbourhood $U$ of $p$ such that $U$ and $\Omega_{E^n_T}$ are homeomorphic.

Let us consider $n$. Observe that every non empty topological space which is $n$-locally Euclidean is also first-countable.

Let us note that every non empty topological space which is 0-locally Euclidean is also discrete and every non empty topological space which is discrete is also 0-locally Euclidean.

Let us consider $n$. One can verify that $E^n_T$ is second-countable.

Let us consider $n$. Note that there exists a non empty topological space which is second-countable, Hausdorff, and $n$-locally Euclidean.

Let us consider $n$, $M$. We say that $M$ is $n$-manifold if and only if:

(Def. 5) $M$ is second-countable, Hausdorff, and $n$-locally Euclidean.

Let us consider $M$. We say that $M$ is manifold-like if and only if:

(Def. 6) There exists $n$ such that $M$ is $n$-manifold.

Let us consider $n$. Observe that there exists a non empty topological space which is $n$-manifold.

Let us consider $n$. One can check the following observations:

- every non empty topological space which is $n$-manifold is also second-countable, Hausdorff, and $n$-locally Euclidean,
- every non empty topological space which is second-countable, Hausdorff, and $n$-locally Euclidean is also $n$-manifold, and
- every non empty topological space which is $n$-manifold is also manifold-like.

Let us note that every non empty topological space which is second-countable and discrete is also 0-manifold.
Let us consider $n$ and let $M$ be an $n$-manifold non empty topological space. One can verify that every non empty subspace of $M$ which is open is also $n$-manifold.

Let us note that there exists a non empty topological space which is manifold-like.

A manifold is a manifold-like non empty topological space.

References


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