

## Difference and Difference Quotient. Part IV

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**Summary.** In this article, we give some important theorems of forward difference, backward difference, central difference and difference quotient and forward difference, backward difference, central difference and difference quotient formulas of some special functions.

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The papers [2], [7], [13], [3], [1], [6], [9], [4], [14], [8], [5], [15], [11], [12], and [10] provide the notation and terminology for this paper.

We adopt the following rules:  $n$  denotes an element of  $\mathbb{N}$ ,  $h, k, x, x_0, x_1, x_2, x_3$  denote real numbers, and  $f, g$  denote functions from  $\mathbb{R}$  into  $\mathbb{R}$ .

Next we state a number of propositions:

- (1) If  $x_0 > 0$  and  $x_1 > 0$ , then  $\log_e x_0 - \log_e x_1 = \log_e(\frac{x_0}{x_1})$ .
- (2) If  $x_0 > 0$  and  $x_1 > 0$ , then  $\log_e x_0 + \log_e x_1 = \log_e(x_0 \cdot x_1)$ .
- (3) If  $x > 0$ , then  $\log_e x = (\text{the function } \ln)(x)$ .
- (4) If  $x_0 > 0$  and  $x_1 > 0$ , then  $(\text{the function } \ln)(x_0) - (\text{the function } \ln)(x_1) = (\text{the function } \ln)(\frac{x_0}{x_1})$ .
- (5) Suppose for every  $x$  holds  $f(x) = \frac{k}{x^2}$  and  $x_0 \neq 0$  and  $x_1 \neq 0$  and  $x_2 \neq 0$  and  $x_3 \neq 0$  and  $x_0, x_1, x_2, x_3$  are mutually different. Then 
$$\Delta[f](x_0, x_1, x_2, x_3) = \frac{k \cdot (\frac{1}{x_1 \cdot x_2 \cdot x_0} \cdot (\frac{1}{x_0} + \frac{1}{x_2} + \frac{1}{x_1}) - \frac{1}{x_2 \cdot x_1 \cdot x_3} \cdot (\frac{1}{x_3} + \frac{1}{x_1} + \frac{1}{x_2}))}{x_0 - x_3}$$
.

- (6) Suppose  $x_0 \in \text{dom}(\text{the function cot})$  and  $x_1 \in \text{dom}(\text{the function cot})$ . Then  $\Delta[(\text{the function cot}) (\text{the function cot})](x_0, x_1) = \frac{(\cos x_1)^2 - (\cos x_0)^2}{(\sin x_0 \cdot \sin x_1)^2 \cdot (x_0 - x_1)}$ .
- (7) Suppose  $x \in \text{dom}(\text{the function cot})$  and  $x + h \in \text{dom}(\text{the function cot})$ . Then  $(\Delta_h[(\text{the function cot}) (\text{the function cot})])(x) = \frac{\frac{1}{2} \cdot (\cos(2 \cdot (x+h)) - \cos(2 \cdot x))}{(\sin(x+h) \cdot \sin x)^2}$ .
- (8) Suppose  $x \in \text{dom}(\text{the function cot})$  and  $x - h \in \text{dom}(\text{the function cot})$ . Then  $(\nabla_h[(\text{the function cot}) (\text{the function cot})])(x) = \frac{\frac{1}{2} \cdot (\cos(2 \cdot x) - \cos(2 \cdot (h-x)))}{(\sin x \cdot \sin(x-h))^2}$ .
- (9) Suppose  $x + \frac{h}{2} \in \text{dom}(\text{the function cot})$  and  $x - \frac{h}{2} \in \text{dom}(\text{the function cot})$ . Then  $(\delta_h[(\text{the function cot}) (\text{the function cot})])(x) = \frac{\frac{1}{2} \cdot (\cos(h+2 \cdot x) - \cos(h-2 \cdot x))}{(\sin(x+\frac{h}{2}) \cdot \sin(x-\frac{h}{2}))^2}$ .
- (10) If  $x_0, x_1 \in \text{dom cosec}$ , then  $\Delta[\text{cosec cosec}](x_0, x_1) = \frac{4 \cdot (\sin(x_1+x_0) \cdot \sin(x_1-x_0))}{(\cos(x_0+x_1) - \cos(x_0-x_1))^2 \cdot (x_0-x_1)}$ .
- (11) If  $x, x+h \in \text{dom cosec}$ , then  $(\Delta_h[\text{cosec cosec}])(x) = -\frac{4 \cdot \sin(2 \cdot x+h) \cdot \sin h}{(\cos(2 \cdot x+h) - \cos h)^2}$ .
- (12) If  $x, x-h \in \text{dom cosec}$ , then  $(\nabla_h[\text{cosec cosec}])(x) = -\frac{4 \cdot \sin(2 \cdot x-h) \cdot \sin h}{(\cos(2 \cdot x-h) - \cos h)^2}$ .
- (13) If  $x + \frac{h}{2}, x - \frac{h}{2} \in \text{dom cosec}$ , then  $(\delta_h[\text{cosec cosec}])(x) = -\frac{4 \cdot \sin(2 \cdot x) \cdot \sin h}{(\cos(2 \cdot x) - \cos h)^2}$ .
- (14) If  $x_0, x_1 \in \text{dom sec}$ , then  $\Delta[\text{sec sec}](x_0, x_1) = \frac{4 \cdot (\sin(x_0+x_1) \cdot \sin(x_0-x_1))}{(\cos(x_0+x_1) + \cos(x_0-x_1))^2 \cdot (x_0-x_1)}$ .
- (15) If  $x, x+h \in \text{dom sec}$ , then  $(\Delta_h[\text{sec sec}])(x) = \frac{4 \cdot \sin(2 \cdot x+h) \cdot \sin h}{(\cos(2 \cdot x+h) + \cos h)^2}$ .
- (16) If  $x, x-h \in \text{dom sec}$ , then  $(\nabla_h[\text{sec sec}])(x) = \frac{4 \cdot \sin(2 \cdot x-h) \cdot \sin h}{(\cos(2 \cdot x-h) + \cos h)^2}$ .
- (17) If  $x + \frac{h}{2}, x - \frac{h}{2} \in \text{dom sec}$ , then  $(\delta_h[\text{sec sec}])(x) = \frac{4 \cdot \sin(2 \cdot x) \cdot \sin h}{(\cos(2 \cdot x) + \cos h)^2}$ .
- (18) If  $x_0, x_1 \in \text{dom cosec} \cap \text{dom sec}$ , then  $\Delta[\text{cosec sec}](x_0, x_1) = \frac{4 \cdot (\cos(x_1+x_0) \cdot \sin(x_1-x_0))}{\frac{\sin(2 \cdot x_0) \cdot \sin(2 \cdot x_1)}{x_0-x_1}}$ .
- (19) If  $x+h, x \in \text{dom cosec} \cap \text{dom sec}$ , then  $(\Delta_h[\text{cosec sec}])(x) = -4 \cdot \frac{\cos(2 \cdot x+h) \cdot \sin h}{\sin(2 \cdot (x+h)) \cdot \sin(2 \cdot x)}$ .
- (20) If  $x-h, x \in \text{dom cosec} \cap \text{dom sec}$ , then  $(\nabla_h[\text{cosec sec}])(x) = -4 \cdot \frac{\cos(2 \cdot x-h) \cdot \sin h}{\sin(2 \cdot x) \cdot \sin(2 \cdot (x-h))}$ .
- (21) If  $x + \frac{h}{2}, x - \frac{h}{2} \in \text{dom cosec} \cap \text{dom sec}$ , then  $(\delta_h[\text{cosec sec}])(x) = -4 \cdot \frac{\cos(2 \cdot x) \cdot \sin h}{\sin(2 \cdot x+h) \cdot \sin(2 \cdot x-h)}$ .
- (22) Suppose  $x_0 \in \text{dom}(\text{the function tan})$  and  $x_1 \in \text{dom}(\text{the function tan})$ . Then  $\Delta[(\text{the function tan}) (\text{the function tan}) (\text{the function cos})](x_0, x_1) = \Delta[(\text{the function tan}) (\text{the function sin})](x_0, x_1)$ .
- (23) Suppose  $x \in \text{dom}(\text{the function tan})$  and  $x+h \in \text{dom}(\text{the function tan})$ . Then  $(\Delta_h[(\text{the function tan}) (\text{the function tan}) (\text{the function cos})])(x) =$

- ((the function tan) (the function sin))(x + h) - ((the function tan) (the function sin))(x).
- (24) Suppose  $x \in \text{dom}(\text{the function tan})$  and  $x - h \in \text{dom}(\text{the function tan})$ . Then  $(\nabla_h[(\text{the function tan}) (\text{the function tan}) (\text{the function cos})])(x) = ((\text{the function tan}) (\text{the function sin}))(x) - ((\text{the function tan}) (\text{the function sin}))(x - h)$ .
- (25) Suppose  $x + \frac{h}{2} \in \text{dom}(\text{the function tan})$  and  $x - \frac{h}{2} \in \text{dom}(\text{the function tan})$ . Then  $(\delta_h[(\text{the function tan}) (\text{the function tan}) (\text{the function cos})])(x) = ((\text{the function tan}) (\text{the function sin}))(x + \frac{h}{2}) - ((\text{the function tan}) (\text{the function sin}))(x - \frac{h}{2})$ .
- (26) Suppose  $x_0 \in \text{dom}(\text{the function cot})$  and  $x_1 \in \text{dom}(\text{the function cot})$ . Then  $\Delta[(\text{the function cot}) (\text{the function cot}) (\text{the function sin})](x_0, x_1) = \Delta[(\text{the function cot}) (\text{the function cos})](x_0, x_1)$ .
- (27) Suppose  $x \in \text{dom}(\text{the function cot})$  and  $x + h \in \text{dom}(\text{the function cot})$ . Then  $(\Delta_h[(\text{the function cot}) (\text{the function cot}) (\text{the function sin})])(x) = ((\text{the function cot}) (\text{the function cos}))(x + h) - ((\text{the function cot}) (\text{the function cos}))(x)$ .
- (28) Suppose  $x \in \text{dom}(\text{the function cot})$  and  $x - h \in \text{dom}(\text{the function cot})$ . Then  $(\nabla_h[(\text{the function cot}) (\text{the function cot}) (\text{the function sin})])(x) = ((\text{the function cot}) (\text{the function cos}))(x) - ((\text{the function cot}) (\text{the function cos}))(x - h)$ .
- (29) Suppose  $x + \frac{h}{2} \in \text{dom}(\text{the function cot})$  and  $x - \frac{h}{2} \in \text{dom}(\text{the function cot})$ . Then  $(\delta_h[(\text{the function cot}) (\text{the function cot}) (\text{the function sin})])(x) = ((\text{the function cot}) (\text{the function cos}))(x + \frac{h}{2}) - ((\text{the function cot}) (\text{the function cos}))(x - \frac{h}{2})$ .
- (30) If  $x_0 > 0$  and  $x_1 > 0$ , then  $\Delta[\text{the function ln}](x_0, x_1) = \frac{(\text{the function ln})(\frac{x_0}{x_1})}{x_0 - x_1}$ .
- (31) If  $x > 0$  and  $x + h > 0$ , then  $(\Delta_h[\text{the function ln}])(x) = (\text{the function ln})(1 + \frac{h}{x})$ .
- (32) If  $x > 0$  and  $x - h > 0$ , then  $(\nabla_h[\text{the function ln}])(x) = (\text{the function ln})(1 + \frac{h}{x-h})$ .
- (33) If  $x + \frac{h}{2} > 0$  and  $x - \frac{h}{2} > 0$ , then  $(\delta_h[\text{the function ln}])(x) = (\text{the function ln})(1 + \frac{h}{x - \frac{h}{2}})$ .
- (34) For all real numbers  $h, k$  holds  $\exp(h - k) = \frac{\exp h}{\exp k}$ .
- (35)  $(\Delta_h[f])(x) = (\text{Shift}(f, h))(x) - f(x)$ .
- (36) If for every  $x$  holds  $f(x) = (\Delta_h[g])(x)$ , then  $\Delta[f](x_0, x_1) = \Delta[g](x_0 + h, x_1 + h) - \Delta[g](x_0, x_1)$ .
- (37)  $(\Delta_h[\Delta_h[f]])(x) = (\Delta_{2 \cdot h}[f])(x) - 2 \cdot (\Delta_h[f])(x)$ .
- (38)  $(\nabla_h[\Delta_h[f]])(x) = (\Delta_h[f])(x) - (\nabla_h[f])(x)$ .

- (39)  $(\delta_h[\Delta_h[f]])(x) = (\Delta_h[f])(x + \frac{h}{2}) - (\delta_h[f])(x)$ .
- (40)  $(\vec{\Delta}_h[f])(1)(x) = (\vec{\Delta}_h[f])(0)(x + h) - (\vec{\Delta}_h[f])(0)(x)$ .
- (41)  $(\vec{\Delta}_h[f])(n + 1)(x) = (\vec{\Delta}_h[f])(n)(x + h) - (\vec{\Delta}_h[f])(n)(x)$ .
- (42)  $(\nabla_h[f])(x) = f(x) - (\text{Shift}(f, -h))(x)$ .
- (43) If for every  $x$  holds  $f(x) = (\nabla_h[g])(x)$ , then  $\Delta[f](x_0, x_1) = \Delta[g](x_0, x_1) - \Delta[g](x_0 - h, x_1 - h)$ .
- (44)  $(\Delta_h[\nabla_h[f]])(x) = (\Delta_h[f])(x) - (\nabla_h[f])(x)$ .
- (45)  $(\nabla_h[\nabla_h[f]])(x) = 2 \cdot (\nabla_h[f])(x) - (\nabla_{2 \cdot h}[f])(x)$ .
- (46)  $(\delta_h[\nabla_h[f]])(x) = (\delta_h[f])(x) - (\nabla_h[f])(x - \frac{h}{2})$ .
- (47)  $(\vec{\nabla}_h[f])(1)(x) = (\vec{\nabla}_h[f])(0)(x) - (\vec{\nabla}_h[f])(0)(x - h)$ .
- (48)  $(\vec{\nabla}_h[f])(n + 1)(x) = (\vec{\nabla}_h[f])(n)(x) - (\vec{\nabla}_h[f])(n)(x - h)$ .
- (49)  $(\delta_h[f])(x) = (\text{Shift}(f, \frac{h}{2}))(x) - (\text{Shift}(f, -\frac{h}{2}))(x)$ .
- (50) If for every  $x$  holds  $f(x) = (\delta_h[g])(x)$ , then  $\Delta[f](x_0, x_1) = \Delta[g](x_0 + \frac{h}{2}, x_1 + \frac{h}{2}) - \Delta[g](x_0 - \frac{h}{2}, x_1 - \frac{h}{2})$ .
- (51)  $(\Delta_h[\delta_h[f]])(x) = (\Delta_h[f])(x + \frac{h}{2}) - (\delta_h[f])(x)$ .
- (52)  $(\nabla_h[\delta_h[f]])(x) = (\delta_h[f])(x) - (\nabla_h[f])(x - \frac{h}{2})$ .
- (53)  $(\delta_h[\delta_h[f]])(x) = (\Delta_h[f])(x) - (\nabla_h[f])(x)$ .
- (54)  $(\vec{\delta}_h[f])(1)(x) = (\vec{\delta}_h[f])(0)(x + \frac{h}{2}) - (\vec{\delta}_h[f])(0)(x - \frac{h}{2})$ .
- (55)  $(\vec{\delta}_h[f])(n + 1)(x) = (\vec{\delta}_h[f])(n)(x + \frac{h}{2}) - (\vec{\delta}_h[f])(n)(x - \frac{h}{2})$ .
- (56) Suppose  $x_0 \in \text{dom}(\text{the function tan})$  and  $x_1 \in \text{dom}(\text{the function tan})$ . Then  $\Delta[(\text{the function tan}) (\text{the function tan}) (\text{the function sin})](x_0, x_1) = \frac{(\sin x_0)^3 \cdot (\cos x_1)^2 - (\sin x_1)^3 \cdot (\cos x_0)^2}{(\cos x_0)^2 \cdot (\cos x_1)^2 \cdot (x_0 - x_1)}$ .
- (57) Suppose  $x \in \text{dom}(\text{the function tan})$  and  $x + h \in \text{dom}(\text{the function tan})$ . Then  $(\Delta_h[(\text{the function tan}) (\text{the function tan}) (\text{the function sin})])(x) = (\text{the function sin})(x + h)^3 \cdot ((\text{the function cos})(x + h)^{-1})^2 - (\text{the function sin})(x)^3 \cdot ((\text{the function cos})(x)^{-1})^2$ .
- (58) Suppose  $x \in \text{dom}(\text{the function tan})$  and  $x - h \in \text{dom}(\text{the function tan})$ . Then  $(\nabla_h[(\text{the function tan}) (\text{the function tan}) (\text{the function sin})])(x) = (\text{the function sin})(x)^3 \cdot ((\text{the function cos})(x)^{-1})^2 - (\text{the function sin})(x - h)^3 \cdot ((\text{the function cos})(x - h)^{-1})^2$ .
- (59) Suppose  $x + \frac{h}{2} \in \text{dom}(\text{the function tan})$  and  $x - \frac{h}{2} \in \text{dom}(\text{the function tan})$ . Then  $(\delta_h[(\text{the function tan}) (\text{the function tan}) (\text{the function sin})])(x) = (\text{the function sin})(x + \frac{h}{2})^3 \cdot ((\text{the function cos})(x + \frac{h}{2})^{-1})^2 - (\text{the function sin})(x - \frac{h}{2})^3 \cdot ((\text{the function cos})(x - \frac{h}{2})^{-1})^2$ .
- (60) Suppose  $x_0 \in \text{dom}(\text{the function cot})$  and  $x_1 \in \text{dom}(\text{the function cot})$ . Then  $\Delta[(\text{the function cot}) (\text{the function cot}) (\text{the function cos})](x_0, x_1) = \frac{(\cos x_0)^3 \cdot (\sin x_1)^2 - (\cos x_1)^3 \cdot (\sin x_0)^2}{(\sin x_0)^2 \cdot (\sin x_1)^2 \cdot (x_0 - x_1)}$ .

- (61) Suppose  $x \in \text{dom}(\text{the function cot})$  and  $x + h \in \text{dom}(\text{the function cot})$ . Then  $(\Delta_h[(\text{the function cot}) (\text{the function cot}) (\text{the function cos})])(x) = (\text{the function cos})(x + h)^3 \cdot ((\text{the function sin})(x + h)^{-1})^2 - (\text{the function cos})(x)^3 \cdot ((\text{the function sin})(x)^{-1})^2$ .
- (62) Suppose  $x \in \text{dom}(\text{the function cot})$  and  $x - h \in \text{dom}(\text{the function cot})$ . Then  $(\nabla_h[(\text{the function cot}) (\text{the function cot}) (\text{the function cos})])(x) = (\text{the function cos})(x)^3 \cdot ((\text{the function sin})(x)^{-1})^2 - (\text{the function cos})(x - h)^3 \cdot ((\text{the function sin})(x - h)^{-1})^2$ .
- (63) Suppose  $x + \frac{h}{2} \in \text{dom}(\text{the function cot})$  and  $x - \frac{h}{2} \in \text{dom}(\text{the function cot})$ . Then  $(\delta_h[(\text{the function cot}) (\text{the function cot}) (\text{the function cos})])(x) = (\text{the function cos})(x + \frac{h}{2})^3 \cdot ((\text{the function sin})(x + \frac{h}{2})^{-1})^2 - (\text{the function cos})(x - \frac{h}{2})^3 \cdot ((\text{the function sin})(x - \frac{h}{2})^{-1})^2$ .

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