The Mycielskian of a Graph\(^1\)

Piotr Rudnicki  
University of Alberta  
Edmonton, Canada

Lorna Stewart  
University of Alberta  
Edmonton, Canada

Summary. Let \(\omega(G)\) and \(\chi(G)\) be the clique number and the chromatic number of a graph \(G\). Mycielski [11] presented a construction that for any \(n\) creates a graph \(M_n\) which is triangle-free (\(\omega(G) = 2\)) with \(\chi(G) > n\). The starting point is the complete graph of two vertices (\(K_2\)). \(M_{n+1}\) is obtained from \(M_n\) through the operation \(\mu(G)\) called the Mycielskian of a graph \(G\).

We first define the operation \(\mu(G)\) and then show that \(\omega(\mu(G)) = \omega(G)\) and \(\chi(\mu(G)) = \chi(G) + 1\). This is done for arbitrary graph \(G\), see also [10]. Then we define the sequence of graphs \(M_n\) each of exponential size in \(n\) and give their clique and chromatic numbers.

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The notation and terminology used here have been introduced in the following papers: [1], [15], [13], [8], [5], [2], [14], [9], [16], [3], [6], [18], [19], [12], [17], [4], and [7].

1. Preliminaries

One can prove the following propositions:

1. For all real numbers \(x, y, z\) such that \(0 \leq x\) holds \(x \cdot (y - z) = x \cdot y - x \cdot z\).
2. For all natural numbers \(x, y, z\) holds \(x \in y \setminus z\) iff \(z \leq x < y\).
3. For all sets \(A, B, C, D, E, X\) such that \(X \subseteq A\) or \(X \subseteq B\) or \(X \subseteq C\) or \(X \subseteq D\) or \(X \subseteq E\) holds \(X \subseteq A \cup B \cup C \cup D \cup E\).
4. For all sets \(A, B, C, D, E, x\) holds \(x \in A \cup B \cup C \cup D \cup E\) iff \(x \in A\) or \(x \in B\) or \(x \in C\) or \(x \in D\) or \(x \in E\).

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(5) Let $R$ be a symmetric relational structure and $x$, $y$ be sets. Suppose $x \in$ the carrier of $R$ and $y \in$ the carrier of $R$ and $\langle x, y \rangle \in$ the internal relation of $R$. Then $\langle y, x \rangle \in$ the internal relation of $R$.

(6) For every symmetric relational structure $R$ and for all elements $x$, $y$ of $R$ such that $x \leq y$ holds $y \leq x$.

2. Partitions

One can prove the following proposition

(7) For every set $X$ and for every partition $P$ of $X$ holds $\overline{P} \subseteq \overline{X}$.

Let $X$ be a set, let $P$ be a partition of $X$, and let $S$ be a subset of $X$. The functor $P|S$ yields a partition of $S$ and is defined by:

(Def. 1) $P|S = \{ x \cap S; x \text{ ranges over elements of } P; x \text{ meets } S \}.$

Let $X$ be a set. Observe that there exists a partition of $X$ which is finite.

Let $X$ be a set, let $P$ be a finite partition of $X$, and let $S$ be a subset of $X$. Observe that $P|S$ is finite.

One can prove the following propositions:

(8) For every set $X$ and for every finite partition $P$ of $X$ and for every subset $S$ of $X$ holds $\overline{P|S} \leq \overline{P}$.

(9) Let $X$ be a set, $P$ be a finite partition of $X$, and $S$ be a subset of $X$. Then for every set $p$ such that $p \in P$ holds $p$ meets $S$ if and only if $\overline{P|S} = \overline{P}$.

(10) Let $R$ be a relational structure, $C$ be a coloring of $R$, and $S$ be a subset of $R$. Then $C|S$ is a coloring of sub($S$).

3. Chromatic Number and Clique Cover Number

Let $R$ be a relational structure. We say that $R$ is finitely colorable if and only if:

(Def. 2) There exists a coloring of $R$ which is finite.

One can check that there exists a relational structure which is finitely colorable.

Let us observe that every relational structure which is finite is also finitely colorable.

Let $R$ be a finitely colorable relational structure. Observe that there exists a coloring of $R$ which is finite.

Let $R$ be a finitely colorable relational structure and let $S$ be a subset of $R$. One can verify that sub($S$) is finitely colorable.

Let $R$ be a finitely colorable relational structure. The functor $\chi(R)$ yielding a natural number is defined by:
(Def. 3) There exists a finite coloring $C$ of $R$ such that $\overline{C} = \chi(R)$ and for every finite coloring $C$ of $R$ holds $\chi(R) \leq \overline{C}$.

Let $R$ be an empty relational structure. Observe that $\chi(R)$ is empty.

Let $R$ be a non empty finitely colorable relational structure. Observe that $\chi(R)$ is positive.

Let $R$ be a relational structure. We say that $R$ has finite clique cover if and only if:

(Def. 4) There exists a clique-partition of $R$ which is finite.

One can verify that there exists a relational structure which has finite clique cover.

One can verify that every relational structure which is finite has also finite clique cover.

Let $R$ be a relational structure with finite clique cover. Observe that there exists a clique-partition of $R$ which is finite.

Let $R$ be a relational structure with finite clique cover and let $S$ be a subset of $R$. Observe that sub($S$) has finite clique cover.

Let $R$ be a relational structure with finite clique cover. The functor $\kappa(R)$ yielding a natural number is defined by:

(Def. 5) There exists a finite clique-partition $C$ of $R$ such that $\overline{C} = \kappa(R)$ and for every finite clique-partition $C$ of $R$ holds $\kappa(R) \leq \overline{C}$.

Let $R$ be an empty relational structure. One can check that $\kappa(R)$ is empty.

Let $R$ be a non empty relational structure with finite clique cover. One can verify that $\kappa(R)$ is positive.

We now state several propositions:

(11) For every finite relational structure $R$ holds $\omega(R) \leq \text{the carrier of } R$.

(12) For every finite relational structure $R$ holds $\alpha(R) \leq \text{the carrier of } R$.

(13) For every finite relational structure $R$ holds $\chi(R) \leq \text{the carrier of } R$.

(14) For every finite relational structure $R$ holds $\kappa(R) \leq \text{the carrier of } R$.

(15) For every finitely colorable relational structure $R$ with finite clique number holds $\omega(R) \leq \chi(R)$.

(16) For every relational structure $R$ with finite stability number and finite clique cover holds $\alpha(R) \leq \kappa(R)$.

4. Complement

The following two propositions are true:

(17) Let $R$ be a relational structure, $x, y$ be elements of $R$, and $a, b$ be elements of ComplRelStr $R$. If $x = a$ and $y = b$ and $x \leq y$, then $a \not\leq b$. 
Let $R$ be a relational structure, $x, y$ be elements of $R$, and $a, b$ be elements of ComplRelStr $R$. If $x = a$ and $y = b$ and $x \neq y$ and $x \in$ the carrier of $R$ and $a \not\leq b$, then $x \leq y$.

Let $R$ be a finite relational structure. Note that ComplRelStr $R$ is finite.

Next we state four propositions:

(19) For every symmetric relational structure $R$ holds every clique of $R$ is a stable set of ComplRelStr $R$.

(20) For every symmetric relational structure $R$ holds every clique of ComplRelStr $R$ is a stable set of $R$.

(21) For every relational structure $R$ holds every stable set of $R$ is a clique of ComplRelStr $R$.

(22) For every relational structure $R$ holds every stable set of ComplRelStr $R$ is a clique of $R$.

Let $R$ be a relational structure with finite clique number. One can verify that ComplRelStr $R$ has finite stability number.

Let $R$ be a symmetric relational structure with finite stability number. Observe that ComplRelStr $R$ has finite clique number.

The following propositions are true:

(23) For every symmetric relational structure $R$ with finite clique number holds $\omega(R) = \alpha(\text{ComplRelStr } R)$.

(24) For every symmetric relational structure $R$ with finite stability number holds $\alpha(R) = \omega(\text{ComplRelStr } R)$.

(25) For every relational structure $R$ holds every coloring of $R$ is a clique-partition of ComplRelStr $R$.

(26) For every symmetric relational structure $R$ holds every clique-partition of ComplRelStr $R$ is a coloring of $R$.

(27) For every symmetric relational structure $R$ holds every clique-partition of $R$ is a coloring of ComplRelStr $R$.

(28) For every relational structure $R$ holds every coloring of ComplRelStr $R$ is a clique-partition of $R$.

Let $R$ be a finitely colorable relational structure. Observe that ComplRelStr $R$ has finite clique cover.

Let $R$ be a symmetric relational structure with finite clique cover. One can check that ComplRelStr $R$ is finitely colorable.

The following propositions are true:

(29) For every finitely colorable symmetric relational structure $R$ holds $\chi(R) = \kappa(\text{ComplRelStr } R)$.

(30) For every symmetric relational structure $R$ with finite clique cover holds $\kappa(R) = \chi(\text{ComplRelStr } R)$. 
5. Adjacent Set

Let $R$ be a relational structure and let $v$ be an element of $R$. The functor $\text{Adjacent}(v)$ yields a subset of $R$ and is defined as follows:

(Def. 6) For every element $x$ of $R$ holds $x \in \text{Adjacent}(v)$ iff $x < v$ or $v < x$.

The following proposition is true

(31) Let $R$ be a finitely colorable relational structure, $C$ be a finite coloring of $R$, and $c$ be a set. Suppose $c \in C$ and $\overline{c} = \chi(R)$. Then there exists an element $v$ of $R$ such that $v \in c$ and for every element $d$ of $C$ such that $d \neq c$ there exists an element $w$ of $R$ such that $w \in \text{Adjacent}(v)$ and $w \in d$.

6. Natural Numbers as Vertices

Let $n$ be a natural number. A strict relational structure is said to be a relational structure of $n$ if:

(Def. 7) The carrier of it = $n$.

Let us observe that every relational structure of 0 is empty.

Let $n$ be a non empty natural number. Note that every relational structure of $n$ is non empty.

Let $n$ be a natural number. Note that every relational structure of $n$ is finite and there exists a relational structure of $n$ which is irreflexive.

Let $n$ be a natural number. The functor $K(n)$ yields a relational structure of $n$ and is defined as follows:

(Def. 8) The internal relation of $K(n) = n \times n \setminus \text{id}_n$.

The following proposition is true

(32) Let $n$ be a natural number and $x$, $y$ be sets. Suppose $x$, $y \in n$. Then $\langle x, y \rangle \in$ the internal relation of $K(n)$ if and only if $x \neq y$.

Let $n$ be a natural number. Note that $K(n)$ is irreflexive and symmetric.

Let $n$ be a natural number. Observe that $\Omega_{K(n)}$ is a clique.

The following propositions are true:

(33) For every natural number $n$ holds $\omega(K(n)) = n$.

(34) For every non empty natural number $n$ holds $\alpha(K(n)) = 1$.

(35) For every natural number $n$ holds $\chi(K(n)) = n$.

(36) For every non empty natural number $n$ holds $\kappa(K(n)) = 1$. 
7. Mycielskian of a Graph

Let \( n \) be a natural number and let \( R \) be a relational structure of \( n \). The functor Mycielskian \( R \) yields a relational structure of \( 2 \cdot n + 1 \) and is defined by the condition (Def. 9).

(Def. 9) The internal relation of Mycielskian \( R = \) (the internal relation of \( R \)) \( \cup \) \{ \( \langle \langle x, y + n \rangle \rangle \); \( x \) ranges over elements of \( \mathbb{N} \); \( y \) ranges over elements of \( \mathbb{N} \); \( \langle \langle x, y \rangle \rangle \in \) the internal relation of \( R \) \} \( \cup \) \{ \( \langle \langle x + n, y \rangle \rangle \); \( x \) ranges over elements of \( \mathbb{N} \); \( y \) ranges over elements of \( \mathbb{N} \); \( \langle \langle x, y \rangle \rangle \in \) the internal relation of \( R \) \} \( \cup \{2 \cdot n\} \times \langle n \rangle \) \( \cup \) \{ \( \langle \langle x, y \rangle \rangle \); \( x \) ranges over elements of \( \mathbb{N} \); \( y \) ranges over elements of \( \mathbb{N} \); \( \langle \langle x, y \rangle \rangle \in \) the internal relation of \( R \) \} \( \cup \{2 \cdot n\} \times \{2 \cdot n\} \) \( \cup \) \{ \( \langle \langle x, y \rangle \rangle \); \( x \) ranges over elements of \( \mathbb{N} \); \( y \) ranges over elements of \( \mathbb{N} \); \( \langle \langle x, y \rangle \rangle \in \) the internal relation of \( R \) \} \( \cup \{2 \cdot n\} \times \{2 \cdot n\} \).

One can prove the following propositions:

(37) Let \( n \) be a natural number and \( R \) be a relational structure of \( n \). Then \( \{n\} = \{2 \cdot n\} \subseteq \) the carrier of Mycielskian \( R \).

(38) Let \( n \) be a natural number, \( R \) be a relational structure of \( n \), and \( x, y \) be natural numbers. Suppose \( \langle \langle x, y \rangle \rangle \in \) the internal relation of Mycielskian \( R \). Then

\( i \) \( x < n \) and \( y < n \), or
\( ii \) \( x < n \leq y < 2 \cdot n \), or
\( iii \) \( n \leq x < 2 \cdot n \) and \( y < n \), or
\( iv \) \( x = 2 \cdot n \) and \( n \leq y < 2 \cdot n \), or
\( v \) \( n \leq x < 2 \cdot n \) and \( y = 2 \cdot n \).

(39) Let \( n \) be a natural number and \( R \) be a relational structure of \( n \). Then \( \{n\} \subseteq \) the internal relation of Mycielskian \( R \).

(40) Let \( n \) be a natural number, \( R \) be a relational structure of \( n \), and \( x, y \) be sets. Suppose \( x, y \in n \) and \( \langle \langle x, y \rangle \rangle \in \) the internal relation of Mycielskian \( R \). Then \( \{x, y\} \subseteq \) the internal relation of \( R \).

(41) Let \( n \) be a natural number, \( R \) be a relational structure of \( n \), and \( x, y \) be natural numbers. Suppose \( \langle \langle x, y \rangle \rangle \in \) the internal relation of \( R \). Then \( \langle \langle x, y + n \rangle \rangle \in \) the internal relation of Mycielskian \( R \) and \( \langle \langle x + n, y \rangle \rangle \in \) the internal relation of Mycielskian \( R \).

(42) Let \( n \) be a natural number, \( R \) be a relational structure of \( n \), and \( x, y \) be natural numbers. Suppose \( x \in n \) and \( \langle \langle x, y + n \rangle \rangle \in \) the internal relation of Mycielskian \( R \). Then \( \{x, y\} \subseteq \) the internal relation of \( R \).

(43) Let \( n \) be a natural number, \( R \) be a relational structure of \( n \), and \( x, y \) be natural numbers. Suppose \( y \in n \) and \( \langle \langle x + n, y \rangle \rangle \in \) the internal relation of Mycielskian \( R \). Then \( \{x, y\} \subseteq \) the internal relation of \( R \).

(44) Let \( n \) be a natural number, \( R \) be a relational structure of \( n \), and \( m \) be a natural number. Suppose \( n \leq m < 2 \cdot n \). Then \( \{m, 2 \cdot n\} \subseteq \) the internal relation of Mycielskian \( R \) and \( \{2 \cdot n, m\} \subseteq \) the internal relation of Mycielskian \( R \).
(45) Let \( n \) be a natural number, \( R \) be a relational structure of \( n \), and \( S \) be a subset of Mycielskian \( R \). If \( S = n \), then \( R = \text{sub}(S) \).

(46) For every natural number \( n \) and for every irreflexive relational structure \( R \) of \( n \) such that \( 2 \leq \omega(R) \) holds \( \omega(R) = \omega(\text{Mycielskian} \ R) \).

(47) For every finitely colorable relational structure \( R \) and for every subset \( S \) of \( R \) holds \( \chi(R) \geq \chi(\text{sub}(S)) \).

(48) For every natural number \( n \) and for every irreflexive relational structure \( R \) of \( n \) holds \( \chi(\text{Mycielskian} \ R) = 1 + \chi(R) \).

Let \( n \) be a natural number. The functor Mycielskian \( n \) yielding a relational structure of \( 3 \cdot 2^n - 1 \) is defined by the condition (Def. 10).

(Def. 10) There exists a function \( m_1 \) such that

(i) Mycielskian \( n = m_1(n) \),

(ii) \( \text{dom} m_1 = \mathbb{N} \),

(iii) \( m_1(0) = \mathcal{K}(2) \), and

(iv) for every natural number \( k \) and for every relational structure \( R \) of \( 3 \cdot 2^k - 1 \) such that \( R = m_1(k) \) holds \( m_1(k + 1) = \text{Mycielskian} \ R \).

The following proposition is true

(49) Mycielskian \( 0 = \mathcal{K}(2) \) and for every natural number \( k \) holds \( \text{Mycielskian}(k + 1) = \text{Mycielskian} \ \text{Mycielskian} \ k \).

Let \( n \) be a natural number. One can verify that Mycielskian \( n \) is irreflexive.

Let \( n \) be a natural number. Observe that Mycielskian \( n \) is symmetric.

We now state three propositions:

(50) For every natural number \( n \) holds \( \omega(\text{Mycielskian} \ n) = 2 \) and \( \chi(\text{Mycielskian} \ n) = n + 2 \).

(51) For every natural number \( n \) there exists a finite relational structure \( R \) such that \( \omega(R) = 2 \) and \( \chi(R) > n \).

(52) For every natural number \( n \) there exists a finite relational structure \( R \) such that \( \alpha(R) = 2 \) and \( \kappa(R) > n \).

References


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