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# Normal Subgroup of Product of Groups

Hiroyuki Okazaki
Shinshu University
Nagano, Japan

Kenichi Arai
Shinshu University
Nagano, Japan

Yasunari Shidama
Shinshu University
Nagano, Japan

**Summary.** In [6] it was formalized that the direct product of a family of groups gives a new group. In this article, we formalize that for all $j \in I$, the group $G = \Pi_{i \in I} G_i$ has a normal subgroup isomorphic to $G_j$. Moreover, we show some relations between a family of groups and its direct product.

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The papers [2], [4], [5], [3], [8], [9], [7], [10], [11], [6], [1], [13], and [12] provide the terminology and notation for this paper.

## 1. Normal Subgroup of Product of Groups

Let $I$ be a non empty set, let $F$ be a group-like multiplicative magma family of $I$, and let $i$ be an element of $I$. Note that $F(i)$ is group-like.

Let $I$ be a non empty set, let $F$ be an associative multiplicative magma family of $I$, and let $i$ be an element of $I$. Observe that $F(i)$ is associative.

Let $I$ be a non empty set, let $F$ be a commutative multiplicative magma family of $I$, and let $i$ be an element of $I$. Note that $F(i)$ is commutative.

In the sequel $I$ is a non empty set, $F$ is an associative group-like multiplicative magma family of $I$, and $i$, $j$ are elements of $I$.

We now state the proposition

(1) Let $x$ be a function and $g$ be an element of $F(i)$. Then $\mathrm{dom}\, x = I$ and $x(i) = g$ and for every element $j$ of $I$ such that $j \neq i$ holds $x(j) = \mathbf{1}_{F(j)}$ if and only if $x = \mathbf{1}_{\prod F} +\cdot (i, g)$.

Let $I$ be a non empty set, let $F$ be an associative group-like multiplicative magma family of $I$, and let $i$ be an element of $I$. The functor $\mathrm{ProjSet}(F, i)$ yields a subset of $\prod F$ and is defined by:




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(Def. 1) For every set $x$ holds $x \in \mathrm{ProjSet}(F, i)$ iff there exists an element $g$ of $F(i)$ such that $x = \mathbf{1}_{\prod F} +\cdot (i, g)$.

Let $I$ be a non empty set, let $F$ be an associative group-like multiplicative magma family of $I$, and let $i$ be an element of $I$. Observe that $\mathrm{ProjSet}(F, i)$ is non empty.

Next we state several propositions:

(2) Let $x_0$ be a set. Then $x_0 \in \mathrm{ProjSet}(F, i)$ if and only if there exists a function $x$ and there exists an element $g$ of $F(i)$ such that $x = x_0$ and $\mathrm{dom}\, x = I$ and $x(i) = g$ and for every element $j$ of $I$ such that $j \neq i$ holds $x(j) = \mathbf{1}_{F(j)}$.

(3) Let $g_1$, $g_2$ be elements of $\prod F$ and $z_1$, $z_2$ be elements of $F(i)$. If $g_1 = \mathbf{1}_{\prod F} +\cdot (i, z_1)$ and $g_2 = \mathbf{1}_{\prod F} +\cdot (i, z_2)$, then $g_1 \cdot g_2 = \mathbf{1}_{\prod F} +\cdot (i, z_1 \cdot z_2)$.

(4) For every element $g_1$ of $\prod F$ and for every element $z_1$ of $F(i)$ such that $g_1 = \mathbf{1}_{\prod F} +\cdot (i, z_1)$ holds $g_1{}^{-1} = \mathbf{1}_{\prod F} +\cdot (i, z_1{}^{-1})$.

(5) For all elements $g_1$, $g_2$ of $\prod F$ such that $g_1$, $g_2 \in \mathrm{ProjSet}(F, i)$ holds $g_1 \cdot g_2 \in \mathrm{ProjSet}(F, i)$.

(6) For every element $g$ of $\prod F$ such that $g \in \mathrm{ProjSet}(F, i)$ holds $g^{-1} \in \mathrm{ProjSet}(F, i)$.

Let $I$ be a non empty set, let $F$ be an associative group-like multiplicative magma family of $I$, and let $i$ be an element of $I$. The functor $\mathrm{ProjGroup}(F, i)$ yields a strict subgroup of $\prod F$ and is defined as follows:

(Def. 2) The carrier of $\mathrm{ProjGroup}(F, i) = \mathrm{ProjSet}(F, i)$.

Let us consider $I$, $F$, $i$. The functor $1\mathrm{ProdHom}(F, i)$ yielding a homomorphism from $F(i)$ to $\mathrm{ProjGroup}(F, i)$ is defined as follows:

(Def. 3) For every element $x$ of $F(i)$ holds $(1\mathrm{ProdHom}(F, i))(x) = \mathbf{1}_{\prod F} +\cdot (i, x)$.

Let us consider $I$, $F$, $i$. Note that $1\mathrm{ProdHom}(F, i)$ is bijective.

Let us consider $I$, $F$, $i$. One can check that $\mathrm{ProjGroup}(F, i)$ is normal.

One can prove the following proposition

(7) For all elements $x$, $y$ of $\prod F$ such that $i \neq j$ and $x \in \mathrm{ProjGroup}(F, i)$ and $y \in \mathrm{ProjGroup}(F, j)$ holds $x \cdot y = y \cdot x$.

## 2. Product of Subgroups of a Group

In the sequel $n$ denotes a non empty natural number.

One can prove the following propositions:

(8) Let $F$ be an associative group-like multiplicative magma family of $\mathrm{Seg}\, n$, $J$ be a natural number, and $G_1$ be a group. Suppose $1 \leq J \leq n$ and $G_1 = F(J)$. Let $x$ be an element of $\prod F$ and $s$ be a finite sequence of elements of $\prod F$. Suppose $\mathrm{len}\, s < J$ and for every element $k$ of $\mathrm{Seg}\, n$

such that $k \in \operatorname{dom} s$ holds $s(k) \in \operatorname{ProjGroup}(F, k)$ and $x = \prod s$. Then $x(J) = \mathbf{1}_{(G_1)}$.

(9) Let $F$ be an associative group-like multiplicative magma family of $\operatorname{Seg} n$, $x$ be an element of $\prod F$, and $s$ be a finite sequence of elements of $\prod F$. Suppose $\operatorname{len} s = n$ and for every element $k$ of $\operatorname{Seg} n$ holds $s(k) \in \operatorname{ProjGroup}(F, k)$ and $x = \prod s$. Let $i$ be a natural number. Suppose $1 \leq i \leq n$. Then there exists an element $s_1$ of $\prod F$ such that $s_1 = s(i)$ and $x(i) = s_1(i)$.

(10) Let $F$ be an associative group-like multiplicative magma family of $\operatorname{Seg} n$, $x$ be an element of $\prod F$, and $s$, $t$ be finite sequences of elements of $\prod F$. Suppose that

(i) $\operatorname{len} s = n$,

(ii) for every element $k$ of $\operatorname{Seg} n$ holds $s(k) \in \operatorname{ProjGroup}(F, k)$,

(iii) $x = \prod s$,

(iv) $\operatorname{len} t = n$,

(v) for every element $k$ of $\operatorname{Seg} n$ holds $t(k) \in \operatorname{ProjGroup}(F, k)$, and

(vi) $x = \prod t$.

Then $s = t$.

(11) Let $F$ be an associative group-like multiplicative magma family of $\operatorname{Seg} n$ and $x$ be an element of $\prod F$. Then there exists a finite sequence $s$ of elements of $\prod F$ such that $\operatorname{len} s = n$ and for every element $k$ of $\operatorname{Seg} n$ holds $s(k) \in \operatorname{ProjGroup}(F, k)$ and $x = \prod s$.

(12) Let $G$ be a commutative group and $F$ be an associative group-like multiplicative magma family of $\operatorname{Seg} n$. Suppose that

(i) for every element $i$ of $\operatorname{Seg} n$ holds $F(i)$ is a subgroup of $G$,

(ii) for every element $x$ of $G$ there exists a finite sequence $s$ of elements of $G$ such that $\operatorname{len} s = n$ and for every element $k$ of $\operatorname{Seg} n$ holds $s(k) \in F(k)$ and $x = \prod s$, and

(iii) for all finite sequences $s$, $t$ of elements of $G$ such that $\operatorname{len} s = n$ and for every element $k$ of $\operatorname{Seg} n$ holds $s(k) \in F(k)$ and $\operatorname{len} t = n$ and for every element $k$ of $\operatorname{Seg} n$ holds $t(k) \in F(k)$ and $\prod s = \prod t$ holds $s = t$.

Then there exists a homomorphism $f$ from $\prod F$ to $G$ such that

(iv) $f$ is bijective, and

(v) for every element $x$ of $\prod F$ there exists a finite sequence $s$ of elements of $G$ such that $\operatorname{len} s = n$ and for every element $k$ of $\operatorname{Seg} n$ holds $s(k) \in F(k)$ and $s = x$ and $f(x) = \prod s$.

(13) Let $G$, $F$ be associative commutative group-like multiplicative magma families of $\operatorname{Seg} n$. Suppose that for every element $k$ of $\operatorname{Seg} n$ holds $F(k) = \operatorname{ProjGroup}(G, k)$. Then there exists a homomorphism $f$ from $\prod F$ to $\prod G$ such that

(i) $f$ is bijective, and

(ii) for every element $x$ of $\prod F$ there exists a finite sequence $s$ of elements of $\prod G$ such that $\operatorname{len} s = n$ and for every element $k$ of $\operatorname{Seg} n$ holds $s(k) \in F(k)$ and $s = x$ and $f(x) = \prod s$.

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