Normal Subgroup of Product of Groups

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Summary. In [6] it was formalized that the direct product of a family of groups gives a new group. In this article, we formalize that for all \( j \in I \), the group \( G = \prod_{i \in I} G_i \) has a normal subgroup isomorphic to \( G_j \). Moreover, we show some relations between a family of groups and its direct product.

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The papers [2], [4], [5], [3], [8], [9], [7], [10], [11], [6], [1], [13], and [12] provide the terminology and notation for this paper.

1. Normal Subgroup of Product of Groups

Let \( I \) be a non empty set, let \( F \) be a group-like multiplicative magma family of \( I \), and let \( i \) be an element of \( I \). Note that \( F(i) \) is group-like.

Let \( I \) be a non empty set, let \( F \) be an associative multiplicative magma family of \( I \), and let \( i \) be an element of \( I \). Observe that \( F(i) \) is associative.

Let \( I \) be a non empty set, let \( F \) be a commutative multiplicative magma family of \( I \), and let \( i \) be an element of \( I \). Note that \( F(i) \) is commutative.

In the sequel \( I \) is a non empty set, \( F \) is an associative group-like multiplicative magma family of \( I \), and \( i, j \) are elements of \( I \).

We now state the proposition

1. Let \( x \) be a function and \( g \) be an element of \( F(i) \). Then \( \text{dom}(x) = I \) and \( x(i) = g \) and for every element \( j \) of \( I \) such that \( j \neq i \) holds \( x(j) = 1_{F(j)} \) if and only if \( x = 1_{\prod F} \cdot (i, g) \).

Let \( I \) be a non empty set, let \( F \) be an associative group-like multiplicative magma family of \( I \), and let \( i \) be an element of \( I \). The functor \( \text{ProjSet}(F, i) \) yields a subset of \( \prod F \) and is defined by:
(Def. 1) For every set $x$ holds $x \in \text{ProjSet}(F, i)$ iff there exists an element $g$ of $F(i)$ such that $x = 1_{\prod F} \cdot (i, g)$.

Let $I$ be a non empty set, let $F$ be an associative group-like multiplicative magma family of $I$, and let $i$ be an element of $I$. Observe that $\text{ProjSet}(F, i)$ is non empty.

Next we state several propositions:

(2) Let $x_0$ be a set. Then $x_0 \in \text{ProjSet}(F, i)$ if and only if there exists a function $x$ and there exists an element $g$ of $F(i)$ such that $x = x_0$ and $\text{dom } x = I$ and $x(i) = g$ and for every element $j$ of $I$ such that $j \neq i$ holds $x(j) = 1_{F(j)}$.

(3) Let $g_1, g_2$ be elements of $\prod F$ and $z_1, z_2$ be elements of $F(i)$. If $g_1 = 1_{\prod F} \cdot (i, z_1)$ and $g_2 = 1_{\prod F} \cdot (i, z_2)$, then $g_1 \cdot g_2 = 1_{\prod F} \cdot (i, z_1 \cdot z_2)$.

(4) For every element $g_1$ of $\prod F$ and for every element $z_1$ of $F(i)$ such that $g_1 = 1_{\prod F} \cdot (i, z_1)$ holds $g_1^{-1} = 1_{\prod F} \cdot (i, z_1^{-1})$.

(5) For all elements $g_1, g_2$ of $\prod F$ such that $g_1, g_2 \in \text{ProjSet}(F, i)$ holds $g_1 \cdot g_2 \in \text{ProjSet}(F, i)$.

(6) For every element $g$ of $\prod F$ such that $g \in \text{ProjSet}(F, i)$ holds $g^{-1} \in \text{ProjSet}(F, i)$.

Let $I$ be a non empty set, let $F$ be an associative group-like multiplicative magma family of $I$, and let $i$ be an element of $I$. The functor $\text{ProjGroup}(F, i)$ yields a strict subgroup of $\prod F$ and is defined as follows:

(Def. 2) The carrier of $\text{ProjGroup}(F, i) = \text{ProjSet}(F, i)$.

Let us consider $I, F, i$. The functor $1\text{ProdHom}(F, i)$ yielding a homomorphism from $F(i)$ to $\text{ProjGroup}(F, i)$ is defined as follows:

(Def. 3) For every element $x$ of $F(i)$ holds $(1\text{ProdHom}(F, i))(x) = 1_{\prod F} \cdot (i, x)$.

Let us consider $I, F, i$. Note that $1\text{ProdHom}(F, i)$ is bijective.

Let us consider $I, F, i$. One can check that $\text{ProjGroup}(F, i)$ is normal.

One can prove the following proposition

(7) For all elements $x, y$ of $\prod F$ such that $i \neq j$ and $x \in \text{ProjGroup}(F, i)$ and $y \in \text{ProjGroup}(F, j)$ holds $x \cdot y = y \cdot x$.

2. Product of Subgroups of a Group

In the sequel $n$ denotes a non empty natural number.

One can prove the following propositions:

(8) Let $F$ be an associative group-like multiplicative magma family of $\text{Seg} n$, $J$ be a natural number, and $G_1$ be a group. Suppose $1 \leq J \leq n$ and $G_1 = F(J)$. Let $x$ be an element of $\prod F$ and $s$ be a finite sequence of elements of $\prod F$. Suppose $\text{len } s < J$ and for every element $k$ of $\text{Seg} n$
such that $k \in \text{dom}\, s$ holds $s(k) \in \text{ProjGroup}(F, k)$ and $x = \prod s$. Then $x(J) = 1_{(G_1)}$.

(9) Let $F$ be an associative group-like multiplicative magma family of Seg $n$, $x$ be an element of $\prod F$, and $s$ be a finite sequence of elements of $\prod F$. Suppose $\text{len}\, s = n$ and for every element $k$ of Seg $n$ holds $s(k) \in \text{ProjGroup}(F, k)$ and $x = \prod s$. Let $i$ be a natural number. Suppose $1 \leq i \leq n$. Then there exists an element $s_1$ of $\prod F$ such that $s_1 = s(i)$ and $x(i) = s_1(i)$.

(10) Let $F$ be an associative group-like multiplicative magma family of Seg $n$, $x$ be an element of $\prod F$, and $s, t$ be finite sequences of elements of $\prod F$. Suppose that

(i) $\text{len}\, s = n$,

(ii) for every element $k$ of Seg $n$ holds $s(k) \in \text{ProjGroup}(F, k)$,

(iii) $x = \prod s$,

(iv) $\text{len}\, t = n$,

(v) for every element $k$ of Seg $n$ holds $t(k) \in \text{ProjGroup}(F, k)$, and

(vi) $x = \prod t$.

Then $s = t$.

(11) Let $F$ be an associative group-like multiplicative magma family of Seg $n$ and $x$ be an element of $\prod F$. Then there exists a finite sequence $s$ of elements of $\prod F$ such that $\text{len}\, s = n$ and for every element $k$ of Seg $n$ holds $s(k) \in \text{ProjGroup}(F, k)$ and $x = \prod s$.

(12) Let $G$ be a commutative group and $F$ be an associative group-like multiplicative magma family of Seg $n$. Suppose that

(i) for every element $i$ of Seg $n$ holds $F(i)$ is a subgroup of $G$,

(ii) for every element $x$ of $G$ there exists a finite sequence $s$ of elements of $G$ such that $\text{len}\, s = n$ and for every element $k$ of Seg $n$ holds $s(k) \in F(k)$ and $x = \prod s$, and

(iii) for all finite sequences $s, t$ of elements of $G$ such that $\text{len}\, s = n$ and for every element $k$ of Seg $n$ holds $s(k) \in F(k)$ and $\text{len}\, t = n$ and for every element $k$ of Seg $n$ holds $t(k) \in F(k)$ and $\prod s = \prod t$ holds $s = t$.

Then there exists a homomorphism $f$ from $\prod F$ to $G$ such that

(iv) $f$ is bijective, and

(v) for every element $x$ of $\prod F$ there exists a finite sequence $s$ of elements of $G$ such that $\text{len}\, s = n$ and for every element $k$ of Seg $n$ holds $s(k) \in F(k)$ and $s = x$ and $f(x) = \prod s$.

(13) Let $G, F$ be associative commutative group-like multiplicative magma families of Seg $n$. Suppose that for every element $k$ of Seg $n$ holds $F(k) = \text{ProjGroup}(G, k)$. Then there exists a homomorphism $f$ from $\prod F$ to $\prod G$ such that

(i) $f$ is bijective, and
(ii) for every element $x$ of $\prod F$ there exists a finite sequence $s$ of elements of $\prod G$ such that $\text{len } s = n$ and for every element $k$ of Seg $n$ holds $s(k) \in F(k)$ and $s = x$ and $f(x) = \prod s$.

References


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