FORMALIZED MATHEMATICS
Vol. 19, No. 1, Pages 17–22, 2011
DOI: 10.2478/v10037-011-0003-8

Riemann Integral of Functions from $\mathbb{R}$ into Real Normed Space

Keiichi Miyajima
Ibaraki University
Faculty of Engineering
Hitachi, Japan

Takahiro Kato
Graduate School of Ibaraki University
Faculty of Engineering
Hitachi, Japan

Yasunari Shidama
Shinshu University
Nagano, Japan

Summary. In this article, we define the Riemann integral on functions from $\mathbb{R}$ into real normed space and prove the linearity of this operator. As a result, the Riemann integration can be applied to a wider range of functions. The proof method follows the [16].

MML identifier: `INTEGR18`, version: `7.11.07 4.156.1112`

The terminology and notation used here have been introduced in the following articles: [2], [3], [4], [5], [7], [10], [8], [9], [1], [14], [6], [13], [15], [11], [19], [17], [12], [18], and [20].

1. Preliminaries

Let X be a real normed space, let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into the carrier of X, and let D be a Division of A. A finite sequence of elements of X is said to be a middle volume of f and D if it satisfies the conditions (Def. 1).

(Def. 1)(i) $\operatorname{len} \mathrm{it} = \operatorname{len} D$, and

(ii) for every natural number i such that $i \in \operatorname{dom} D$ there exists a point c of X such that $c \in \operatorname{rng}(f \restriction \operatorname{divset}(D, i))$ and $\mathrm{it}(i) = \operatorname{vol}(\operatorname{divset}(D, i)) \cdot c$.

ISSN 1426–2630(p), 1898-9934(e)

Let X be a real normed space, let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into the carrier of X, let D be a Division of A, and let F be a middle volume of f and D. The functor middle sum(f, F) yielding a point of X is defined by:

(Def. 2) middle sum$(f, F) = \sum F$.

Let X be a real normed space, let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into the carrier of X, and let T be a division sequence of A. A function from $\mathbb{N}$ into (the carrier of X)* is said to be a middle volume sequence of f and T if:

(Def. 3) For every element k of $\mathbb{N}$ holds it(k) is a middle volume of f and $T(k)$.

Let X be a real normed space, let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into the carrier of X, let T be a division sequence of A, let S be a middle volume sequence of f and T, and let k be an element of $\mathbb{N}$. Then $S(k)$ is a middle volume of f and $T(k)$.

Let X be a real normed space, let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into the carrier of X, let T be a division sequence of A, and let S be a middle volume sequence of f and T. The functor middle sum(f, S) yielding a sequence of X is defined as follows:

(Def. 4) For every element i of $\mathbb{N}$ holds (middle sum$(f, S))(i) =$ middle sum$(f, S(i))$.

2. Definition of Riemann Integral on Functions from $\mathbb{R}$ into Real Normed Space

Let X be a real normed space, let A be a closed-interval subset of $\mathbb{R}$, and let f be a function from A into the carrier of X. We say that f is integrable if and only if the condition (Def. 5) is satisfied.

(Def. 5) There exists a point I of X such that for every division sequence T of A and for every middle volume sequence S of f and T if δ_T is convergent and $\lim(\delta_T) = 0$, then middle sum(f, S) is convergent and lim middle sum$(f, S) = I$.

We now state three propositions:

(1) Let X be a real normed space and R_1, R_2, R_3 be finite sequences of elements of X. If $\operatorname{len} R_1 = \operatorname{len} R_2$ and $R_3 = R_1 + R_2$, then $\sum R_3 = \sum R_1 + \sum R_2$.

(2) Let X be a real normed space and R_1, R_2, R_3 be finite sequences of elements of X. If $\operatorname{len} R_1 = \operatorname{len} R_2$ and $R_3 = R_1 - R_2$, then $\sum R_3 = \sum R_1 - \sum R_2$.

(3) Let X be a real normed space, R_1, R_2 be finite sequences of elements of X, and a be an element of $\mathbb{R}$. If $R_2 = a\, R_1$, then $\sum R_2 = a \cdot \sum R_1$.

Let X be a real normed space, let A be a closed-interval subset of $\mathbb{R}$, and let f be a function from A into the carrier of X. Let us assume that f is integrable. The functor integral f yields a point of X and is defined by the condition (Def. 6).

(Def. 6) Let T be a division sequence of A and S be a middle volume sequence of f and T. If δ_T is convergent and $\lim(\delta_T) = 0$, then middle sum(f, S) is convergent and $\lim$ middle sum$(f, S) =$ integral f.

We now state four propositions:

(4) Let X be a real normed space, A be a closed-interval subset of $\mathbb{R}$, r be a real number, and f, h be functions from A into the carrier of X. If $h = r\,f$ and f is integrable, then h is integrable and integral $h = r \cdot$ integral f.

(5) Let X be a real normed space, A be a closed-interval subset of $\mathbb{R}$, and f, h be functions from A into the carrier of X. If $h = -f$ and f is integrable, then h is integrable and integral $h = -$integral f.

(6) Let X be a real normed space, A be a closed-interval subset of $\mathbb{R}$, and f, g, h be functions from A into the carrier of X. Suppose $h = f + g$ and f is integrable and g is integrable. Then h is integrable and integral $h =$ integral $f +$ integral g.

(7) Let X be a real normed space, A be a closed-interval subset of $\mathbb{R}$, and f, g, h be functions from A into the carrier of X. Suppose $h = f - g$ and f is integrable and g is integrable. Then h is integrable and integral $h =$ integral $f -$ integral g.

Let X be a real normed space, let A be a closed-interval subset of $\mathbb{R}$, and let f be a partial function from $\mathbb{R}$ to the carrier of X. We say that f is integrable on A if and only if:

(Def. 7) There exists a function g from A into the carrier of X such that $g = f \restriction A$ and g is integrable.

Let X be a real normed space, let A be a closed-interval subset of $\mathbb{R}$, and let f be a partial function from $\mathbb{R}$ to the carrier of X. Let us assume that $A \subseteq \operatorname{dom} f$. The functor $\int\limits_A f(x)dx$ yields an element of X and is defined as follows:

(Def. 8) There exists a function g from A into the carrier of X such that $g = f \restriction A$ and $\int\limits_A f(x)dx =$ integral g.

We now state several propositions:

(8) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from $\mathbb{R}$ to the carrier of X, and g be a function from A into the carrier of X. Suppose $f \restriction A = g$. Then f is integrable on A if and only if g is integrable.

(9) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from $\mathbb{R}$ to the carrier of X, and g be a function from A into the carrier of X. If

$A \subseteq \operatorname{dom} f$ and $f \restriction A = g$, then $\int_A f(x)dx = \operatorname{integral} g$.

(10) Let X, Y be non empty sets, V be a real normed space, g, f be partial functions from X to the carrier of V, and g_1, f_1 be partial functions from Y to the carrier of V. If $g = g_1$ and $f = f_1$, then $g_1 + f_1 = g + f$.

(11) Let X, Y be non empty sets, V be a real normed space, g, f be partial functions from X to the carrier of V, and g_1, f_1 be partial functions from Y to the carrier of V. If $g = g_1$ and $f = f_1$, then $g_1 - f_1 = g - f$.

(12) Let r be a real number, X, Y be non empty sets, V be a real normed space, g be a partial function from X to the carrier of V, and g_1 be a partial function from Y to the carrier of V. If $g = g_1$, then $r\, g_1 = r\, g$.

3. Linearity of the Integration Operator

Next we state three propositions:

(13) Let r be a real number, A be a closed-interval subset of $\mathbb{R}$, and f be a partial function from $\mathbb{R}$ to the carrier of X. Suppose $A \subseteq \operatorname{dom} f$ and f is integrable on A. Then $r\, f$ is integrable on A and $\int_A (r\, f)(x)dx = r \cdot \int_A f(x)dx$.

(14) Let A be a closed-interval subset of $\mathbb{R}$ and f_1, f_2 be partial functions from $\mathbb{R}$ to the carrier of X. Suppose f_1 is integrable on A and f_2 is integrable on A and $A \subseteq \operatorname{dom} f_1$ and $A \subseteq \operatorname{dom} f_2$. Then $f_1 + f_2$ is integrable on A and $\int_A (f_1 + f_2)(x)dx = \int_A f_1(x)dx + \int_A f_2(x)dx$.

(15) Let A be a closed-interval subset of $\mathbb{R}$ and f_1, f_2 be partial functions from $\mathbb{R}$ to the carrier of X. Suppose f_1 is integrable on A and f_2 is integrable on A and $A \subseteq \operatorname{dom} f_1$ and $A \subseteq \operatorname{dom} f_2$. Then $f_1 - f_2$ is integrable on A and $\int_A (f_1 - f_2)(x)dx = \int_A f_1(x)dx - \int_A f_2(x)dx$.

Let X be a real normed space, let f be a partial function from $\mathbb{R}$ to the carrier of X, and let a, b be real numbers. The functor $\int_a^b f(x)dx$ yielding an element of X is defined as follows:

(Def. 9) $$\int_a^b f(x)dx = \begin{cases} \int_{[a,b]} f(x)dx, & \text{if } a \leq b, \\ -\int_{[b,a]} f(x)dx, & \text{otherwise.} \end{cases}$$

One can prove the following propositions:

(16) Let f be a partial function from $\mathbb{R}$ to the carrier of X, A be a closed-interval subset of $\mathbb{R}$, and a, b be real numbers. If $A = [a, b]$, then $\int\limits_{A} f(x)dx = \int\limits_{a}^{b} f(x)dx$.

(17) Let f be a partial function from $\mathbb{R}$ to the carrier of X and A be a closed-interval subset of $\mathbb{R}$. If $\mathrm{vol}(A) = 0$ and $A \subseteq \mathrm{dom}\, f$, then f is integrable on A and $\int\limits_{A} f(x)dx = 0_X$.

(18) Let f be a partial function from $\mathbb{R}$ to the carrier of X, A be a closed-interval subset of $\mathbb{R}$, and a, b be real numbers. If $A = [b, a]$ and $A \subseteq \mathrm{dom}\, f$, then $-\int\limits_{A} f(x)dx = \int\limits_{a}^{b} f(x)dx$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[6] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[7] Noboru Endou and Artur Korniłowicz. The definition of the Riemann definite integral and some related lemmas. *Formalized Mathematics*, 8(**1**):93–102, 1999.

[8] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Darboux's theorem. *Formalized Mathematics*, 9(**1**):197–200, 2001.

[9] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Definition of integrability for partial functions from $\mathbb{R}$ to $\mathbb{R}$ and integrability for continuous functions. *Formalized Mathematics*, 9(**2**):281–284, 2001.

[10] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Scalar multiple of Riemann definite integral. *Formalized Mathematics*, 9(**1**):191–196, 2001.

[11] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[12] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.

[13] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[14] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(**1**):111–115, 1991.

[15] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.

[16] Murray R. Spiegel. *Theory and Problems of Vector Analysis*. McGraw-Hill, 1974.

[17] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.

[18] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[19] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.
[20] Hiroshi Yamazaki and Yasunari Shidama. Algebra of vector functions. *Formalized Mathematics*, 3(**2**):171–175, 1992.

Received May 20, 2010