Some Properties of $p$-Groups and Commutative $p$-Groups

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Summary. This article describes some properties of $p$-groups and some properties of commutative $p$-groups.

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The notation and terminology used here have been introduced in the following papers: [7], [4], [8], [6], [10], [9], [11], [5], [1], [3], [2], and [12].

1. $p$-Groups

For simplicity, we use the following convention: $G$ is a group, $a, b$ are elements of $G$, $m, n$ are natural numbers, and $p$ is a prime natural number.

One can prove the following propositions:

1. If for every natural number $r$ holds $n \neq p^r$, then there exists an element $s$ of $\mathbb{N}$ such that $s$ is prime and $s \mid n$ and $s \neq p$.
2. For all natural numbers $n, m$ such that $n \mid p^m$ there exists a natural number $r$ such that $n = p^r$ and $r \leq m$.
3. If $a^n = 1_G$, then $(a^{-1})^n = 1_G$.
4. If $(a^{-1})^n = 1_G$, then $a^n = 1_G$.
5. $\text{ord}(a^{-1}) = \text{ord}(a)$.
6. $\text{ord}(a^b) = \text{ord}(a)$.
7. Let $G$ be a group, $N$ be a subgroup of $G$, and $a, b$ be elements of $G$. Suppose $N$ is normal and $b \in N$. Let given $n$. Then there exists an element $g$ of $G$ such that $g \in N$ and $(a \cdot b)^n = a^n \cdot g$. 
(8) Let $G$ be a group, $N$ be a normal subgroup of $G$, $a$ be an element of $G$, and $S$ be an element of $G/N$. If $S = a \cdot N$, then for every $n$ holds $S^n = a^n \cdot N$.

(9) Let $G$ be a group, $H$ be a subgroup of $G$, and $a$, $b$ be elements of $G$. If $a \cdot H = b \cdot H$, then there exists an element $h$ of $G$ such that $a = b \cdot h$ and $h \in H$.

(10) Let $G$ be a finite group and $N$ be a normal subgroup of $G$. If $N$ is a subgroup of $\text{Z}(G)$ and $G/N$ is cyclic, then $G$ is commutative.

(11) Let $G$ be a finite group and $N$ be a normal subgroup of $G$. If $N = \text{Z}(G)$ and $G/N$ is cyclic, then $G$ is commutative.

(12) For every finite group $G$ and for every subgroup $H$ of $G$ such that $\overline{H} \neq \overline{G}$ there exists an element $a$ of $G$ such that $a \notin H$.

Let $p$ be a natural number, let $G$ be a group, and let $a$ be an element of $G$.
We say that $a$ is $p$-power if and only if:

(Def. 1) There exists a natural number $r$ such that $\text{ord}(a) = p^r$.

We now state the proposition

(13) $1_G$ is $m$-power.

Let us consider $G$, $m$. One can verify that there exists an element of $G$ which is $m$-power.

Let us consider $p$, $G$ and let $a$ be a $p$-power element of $G$. Observe that $a^{-1}$ is $p$-power.

One can prove the following proposition

(14) If $a^b$ is $p$-power, then $a$ is $p$-power.

Let us consider $p$, $G$, $b$ and let $a$ be a $p$-power element of $G$. One can verify that $a^b$ is $p$-power.

Let us consider $p$, let $G$ be a commutative group, and let $a$, $b$ be $p$-power elements of $G$. Observe that $a \cdot b$ is $p$-power.

Let us consider $p$ and let $G$ be a finite $p$-group group. One can verify that every element of $G$ is $p$-power.

The following proposition is true

(15) Let $G$ be a finite group, $H$ be a subgroup of $G$, and $a$ be an element of $G$. If $H$ is $p$-group and $a \in H$, then $a$ is $p$-power.

Let us consider $p$ and let $G$ be a finite $p$-group group. One can verify that every subgroup of $G$ is $p$-power.

We now state the proposition

(16) $\{1\}_G$ is $p$-group.

Let us consider $p$ and let $G$ be a group. Note that there exists a subgroup of $G$ which is $p$-group.
Let us consider $p$, let $G$ be a finite group, let $G_1$ be a $p$-group subgroup of $G$, and let $G_2$ be a subgroup of $G$. One can verify that $G_1 \cap G_2$ is $p$-group and $G_2 \cap G_1$ is $p$-group.

Next we state the proposition

(17) For every finite group $G$ such that every element of $G$ is $p$-power holds $G$ is $p$-group.

Let us consider $p$, let $G$ be a finite $p$-group group, and let $N$ be a normal subgroup of $G$. Note that $G/N$ is $p$-group.

The following four propositions are true:

(18) Let $G$ be a finite group and $N$ be a normal subgroup of $G$. If $N$ is $p$-group and $G/N$ is $p$-group, then $G$ is $p$-group.

(19) Let $G$ be a finite commutative group and $H_1, H_2$ be subgroups of $G$. Suppose $H_1$ is $p$-group and $H_2$ is $p$-group and the carrier of $H = H_1 \cdot H_2$. Then $H$ is $p$-group.

(20) Let $G$ be a finite group and $H, N$ be subgroups of $G$. Suppose $N$ is a normal subgroup of $G$ and $H$ is $p$-group and $N$ is $p$-group. Then there exists a strict subgroup $P$ of $G$ such that the carrier of $P = H \cdot N$ and $P$ is $p$-group.

(21) Let $G$ be a finite group and $N_1, N_2$ be normal subgroups of $G$. Suppose $N_1$ is $p$-group and $N_2$ is $p$-group. Then there exists a strict normal subgroup $N$ of $G$ such that the carrier of $N = N_1 \cdot N_2$ and $N$ is $p$-group.

Let us consider $p$, let $G$ be a $p$-group finite group, let $H$ be a finite group, and let $g$ be a homomorphism from $G$ to $H$. Observe that $\text{Im} \ g$ is $p$-group.

The following proposition is true

(22) For all strict groups $G, H$ such that $G$ and $H$ are isomorphic and $G$ is $p$-group holds $\text{expon}(H, p) \leq \text{expon}(G, p)$.

Let $p$ be a prime natural number and let $G$ be a group. Let us assume that $G$ is $p$-group. The functor $\text{expon}(G, p)$ yields a natural number and is defined by:

(Def. 2) $G = p^{\text{expon}(G, p)}$.

Let $p$ be a prime natural number and let $G$ be a group. Then $\text{expon}(G, p)$ is an element of $\mathbb{N}$.

Next we state four propositions:

(23) For every finite group $G$ and for every subgroup $H$ of $G$ such that $G$ is $p$-group holds $\text{expon}(H, p) \leq \text{expon}(G, p)$.

(24) For every strict finite group $G$ such that $G$ is $p$-group and $\text{expon}(G, p) = 0$ holds $G = \{1\}_G$.

(25) For every strict finite group $G$ such that $G$ is $p$-group and $\text{expon}(G, p) = 1$ holds $G$ is cyclic.
(26) Let $G$ be a finite group, $p$ be a prime natural number, and $a$ be an element of $G$. If $G$ is $p$-group and $\text{expon}(G, p) = 2$ and $\text{ord}(a) = p^2$, then $G$ is commutative.

2. Commutative $p$-Groups

Let $p$ be a natural number and let $G$ be a group. We say that $G$ is $p$-commutative group-like if and only if:

(Def. 3) For all elements $a, b$ of $G$ holds $(a \cdot b)^p = a^p \cdot b^p$.

Let $p$ be a natural number and let $G$ be a group. We say that $G$ is $p$-commutative group if and only if:

(Def. 4) $G$ is $p$-group and $p$-commutative group-like.

Let $p$ be a natural number. Observe that every group which is $p$-commutative group is also $p$-group and $p$-commutative group-like and every group which is $p$-group and $p$-commutative group-like is also $p$-commutative group.

The following proposition is true

(27) $\{1\}_G$ is $p$-commutative group-like.

Let us consider $p$. Note that there exists a group which is $p$-commutative group, finite, cyclic, and commutative.

Let us consider $p$ and let $G$ be a $p$-commutative group-like finite group. Note that every subgroup of $G$ is $p$-commutative group-like.

Let us consider $p$. Note that every group which is $p$-group, finite, and commutative is also $p$-commutative group.

We now state the proposition

(28) For every strict finite group $G$ such that $\overline{G} = p$ holds $G$ is $p$-commutative group.

Let us consider $p, G$. One can check that there exists a subgroup of $G$ which is $p$-commutative group and finite.

Let us consider $p$, let $G$ be a finite group, let $H_1$ be a $p$-commutative group-like subgroup of $G$, and let $H_2$ be a subgroup of $G$. One can check that $H_1 \cap H_2$ is $p$-commutative group-like and $H_2 \cap H_1$ is $p$-commutative group-like.

Let us consider $p$, let $G$ be a finite $p$-commutative group-like group, and let $N$ be a normal subgroup of $G$. One can verify that $G/N$ is $p$-commutative group-like.

One can prove the following propositions:

(29) Let $G$ be a finite group and $a, b$ be elements of $G$. Suppose $G$ is $p$-commutative group-like. Let given $n$. Then $(a \cdot b)^{p^n} = a^{p^n} \cdot b^{p^n}$.

(30) Let $G$ be a finite commutative group and $H, H_1, H_2$ be subgroups of $G$. Suppose $H_1$ is $p$-commutative group and $H_2$ is $p$-commutative group and the carrier of $H = H_1 \cdot H_2$. Then $H$ is $p$-commutative group.
(31) Let $G$ be a finite group, $H$ be a subgroup of $G$, and $N$ be a strict normal subgroup of $G$. Suppose $N$ is a subgroup of $Z(G)$ and $H$ is $p$-commutative group and $N$ is $p$-commutative group. Then there exists a strict subgroup $P$ of $G$ such that the carrier of $P = H \cdot N$ and $P$ is $p$-commutative group.

(32) Let $G$ be a finite group and $N_1$, $N_2$ be normal subgroups of $G$. Suppose $N_2$ is a subgroup of $Z(G)$ and $N_1$ is $p$-commutative group and $N_2$ is $p$-commutative group. Then there exists a strict normal subgroup $N$ of $G$ such that the carrier of $N = N_1 \cdot N_2$ and $N$ is $p$-commutative group.

(33) Let $G$, $H$ be groups. Suppose $G$ and $H$ are isomorphic and $G$ is $p$-commutative group-like. Then $H$ is $p$-commutative group-like.

(34) Let $G$, $H$ be strict groups. Suppose $G$ and $H$ are isomorphic and $G$ is $p$-commutative group. Then $H$ is $p$-commutative group.

Let us consider $p$, let $G$ be a $p$-commutative group-like finite group, let $H$ be a finite group, and let $g$ be a homomorphism from $G$ to $H$. Observe that $\text{Im } g$ is $p$-commutative group-like.

The following propositions are true:

(35) For every strict finite group $G$ such that $G$ is $p$-group and $\text{expon}(G,p) = 0$ holds $G$ is $p$-commutative group.

(36) For every strict finite group $G$ such that $G$ is $p$-group and $\text{expon}(G,p) = 1$ holds $G$ is $p$-commutative group.

References


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