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Some Properties of p-Groups and Commutative p-Groups

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Summary. This article describes some properties of p-groups and some properties of commutative p-groups.

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The notation and terminology used here have been introduced in the following papers: [7], [4], [8], [6], [10], [9], [11], [5], [1], [3], [2], and [12].

1. p-Groups

For simplicity, we use the following convention: G is a group, a, b are elements of G, m, n are natural numbers, and p is a prime natural number.

One can prove the following propositions:

(1) If for every natural number r holds $n \neq p^r$, then there exists an element s of $\mathbb{N}$ such that s is prime and $s \mid n$ and $s \neq p$.

(2) For all natural numbers n, m such that $n \mid p^m$ there exists a natural number r such that $n = p^r$ and $r \leq m$.

(3) If $a^n = \mathbf{1}_G$, then $(a^{-1})^n = \mathbf{1}_G$.

(4) If $(a^{-1})^n = \mathbf{1}_G$, then $a^n = \mathbf{1}_G$.

(5) $\mathrm{ord}(a^{-1}) = \mathrm{ord}(a)$.

(6) $\mathrm{ord}(a^b) = \mathrm{ord}(a)$.

(7) Let G be a group, N be a subgroup of G, and a, b be elements of G. Suppose N is normal and $b \in N$. Let given n. Then there exists an element g of G such that $g \in N$ and $(a \cdot b)^n = a^n \cdot g$.

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(8) Let G be a group, N be a normal subgroup of G, a be an element of G, and S be an element of ${}^{G}/_{N}$. If $S = a \cdot N$, then for every n holds $S^n = a^n \cdot N$.

(9) Let G be a group, H be a subgroup of G, and a, b be elements of G. If $a \cdot H = b \cdot H$, then there exists an element h of G such that $a = b \cdot h$ and $h \in H$.

(10) Let G be a finite group and N be a normal subgroup of G. If N is a subgroup of $\mathrm{Z}(G)$ and ${}^{G}/_{N}$ is cyclic, then G is commutative.

(11) Let G be a finite group and N be a normal subgroup of G. If $N = \mathrm{Z}(G)$ and ${}^{G}/_{N}$ is cyclic, then G is commutative.

(12) For every finite group G and for every subgroup H of G such that $\overline{\overline{H}} \neq \overline{\overline{G}}$ there exists an element a of G such that $a \notin H$.

Let p be a natural number, let G be a group, and let a be an element of G. We say that a is p-power if and only if:

(Def. 1) There exists a natural number r such that $\mathrm{ord}(a) = p^r$.

We now state the proposition

(13) $\mathbf{1}_G$ is m-power.

Let us consider G, m. One can verify that there exists an element of G which is m-power.

Let us consider p, G and let a be a p-power element of G. Observe that a^{-1} is p-power.

One can prove the following proposition

(14) If a^b is p-power, then a is p-power.

Let us consider p, G, b and let a be a p-power element of G. One can verify that a^b is p-power.

Let us consider p, let G be a commutative group, and let a, b be p-power elements of G. Observe that $a \cdot b$ is p-power.

Let us consider p and let G be a finite p-group group. One can verify that every element of G is p-power.

The following proposition is true

(15) Let G be a finite group, H be a subgroup of G, and a be an element of G. If H is p-group and $a \in H$, then a is p-power.

Let us consider p and let G be a finite p-group group. One can verify that every subgroup of G is p-group.

We now state the proposition

(16) $\{\mathbf{1}\}_G$ is p-group.

Let us consider p and let G be a group. Note that there exists a subgroup of G which is p-group.

Let us consider p, let G be a finite group, let G_1 be a p-group subgroup of G, and let G_2 be a subgroup of G. One can verify that $G_1 \cap G_2$ is p-group and $G_2 \cap G_1$ is p-group.

Next we state the proposition

(17) For every finite group G such that every element of G is p-power holds G is p-group.

Let us consider p, let G be a finite p-group group, and let N be a normal subgroup of G. Note that ${}^{G}/_{N}$ is p-group.

The following four propositions are true:

(18) Let G be a finite group and N be a normal subgroup of G. If N is p-group and ${}^{G}/_{N}$ is p-group, then G is p-group.

(19) Let G be a finite commutative group and H, H_1, H_2 be subgroups of G. Suppose H_1 is p-group and H_2 is p-group and the carrier of $H = H_1 \cdot H_2$. Then H is p-group.

(20) Let G be a finite group and H, N be subgroups of G. Suppose N is a normal subgroup of G and H is p-group and N is p-group. Then there exists a strict subgroup P of G such that the carrier of $P = H \cdot N$ and P is p-group.

(21) Let G be a finite group and N_1, N_2 be normal subgroups of G. Suppose N_1 is p-group and N_2 is p-group. Then there exists a strict normal subgroup N of G such that the carrier of $N = N_1 \cdot N_2$ and N is p-group.

Let us consider p, let G be a p-group finite group, let H be a finite group, and let g be a homomorphism from G to H. Observe that $\operatorname{Im} g$ is p-group.

The following proposition is true

(22) For all strict groups G, H such that G and H are isomorphic and G is p-group holds H is p-group.

Let p be a prime natural number and let G be a group. Let us assume that G is p-group. The functor $\operatorname{expon}(G, p)$ yields a natural number and is defined by:

(Def. 2) $\overline{\overline{G}} = p^{\operatorname{expon}(G,p)}$.

Let p be a prime natural number and let G be a group. Then $\operatorname{expon}(G, p)$ is an element of $\mathbb{N}$.

Next we state four propositions:

(23) For every finite group G and for every subgroup H of G such that G is p-group holds $\operatorname{expon}(H, p) \leq \operatorname{expon}(G, p)$.

(24) For every strict finite group G such that G is p-group and $\operatorname{expon}(G, p) = 0$ holds $G = \{\mathbf{1}\}_G$.

(25) For every strict finite group G such that G is p-group and $\operatorname{expon}(G, p) = 1$ holds G is cyclic.

(26) Let G be a finite group, p be a prime natural number, and a be an element of G. If G is p-group and $\text{expon}(G, p) = 2$ and $\text{ord}(a) = p^2$, then G is commutative.

2. Commutative p-Groups

Let p be a natural number and let G be a group. We say that G is p-commutative group-like if and only if:

(Def. 3) For all elements a, b of G holds $(a \cdot b)^p = a^p \cdot b^p$.

Let p be a natural number and let G be a group. We say that G is p-commutative group if and only if:

(Def. 4) G is p-group and p-commutative group-like.

Let p be a natural number. Observe that every group which is p-commutative group is also p-group and p-commutative group-like and every group which is p-group and p-commutative group-like is also p-commutative group.

The following proposition is true

(27) $\{\mathbf{1}\}_G$ is p-commutative group-like.

Let us consider p. Note that there exists a group which is p-commutative group, finite, cyclic, and commutative.

Let us consider p and let G be a p-commutative group-like finite group. Note that every subgroup of G is p-commutative group-like.

Let us consider p. Note that every group which is p-group, finite, and commutative is also p-commutative group.

We now state the proposition

(28) For every strict finite group G such that $\overline{\overline{G}} = p$ holds G is p-commutative group.

Let us consider p, G. One can check that there exists a subgroup of G which is p-commutative group and finite.

Let us consider p, let G be a finite group, let H_1 be a p-commutative group-like subgroup of G, and let H_2 be a subgroup of G. One can check that $H_1 \cap H_2$ is p-commutative group-like and $H_2 \cap H_1$ is p-commutative group-like.

Let us consider p, let G be a finite p-commutative group-like group, and let N be a normal subgroup of G. One can verify that ${}^{G}/_{N}$ is p-commutative group-like.

One can prove the following propositions:

(29) Let G be a finite group and a, b be elements of G. Suppose G is p-commutative group-like. Let given n. Then $(a \cdot b)^{p^n} = a^{p^n} \cdot b^{p^n}$.

(30) Let G be a finite commutative group and H, H_1, H_2 be subgroups of G. Suppose H_1 is p-commutative group and H_2 is p-commutative group and the carrier of $H = H_1 \cdot H_2$. Then H is p-commutative group.

(31) Let G be a finite group, H be a subgroup of G, and N be a strict normal subgroup of G. Suppose N is a subgroup of $\mathrm{Z}(G)$ and H is p-commutative group and N is p-commutative group. Then there exists a strict subgroup P of G such that the carrier of $P = H \cdot N$ and P is p-commutative group.

(32) Let G be a finite group and N_1, N_2 be normal subgroups of G. Suppose N_2 is a subgroup of $\mathrm{Z}(G)$ and N_1 is p-commutative group and N_2 is p-commutative group. Then there exists a strict normal subgroup N of G such that the carrier of $N = N_1 \cdot N_2$ and N is p-commutative group.

(33) Let G, H be groups. Suppose G and H are isomorphic and G is p-commutative group-like. Then H is p-commutative group-like.

(34) Let G, H be strict groups. Suppose G and H are isomorphic and G is p-commutative group. Then H is p-commutative group.

Let us consider p, let G be a p-commutative group-like finite group, let H be a finite group, and let g be a homomorphism from G to H. Observe that $\mathrm{Im}\, g$ is p-commutative group-like.

The following propositions are true:

(35) For every strict finite group G such that G is p-group and $\mathrm{expon}(G, p) = 0$ holds G is p-commutative group.

(36) For every strict finite group G such that G is p-group and $\mathrm{expon}(G, p) = 1$ holds G is p-commutative group.

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