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Miscellaneous Facts about Open Functions and Continuous Functions

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Summary. In this article we give definitions of open functions and continuous functions formulated in terms of "balls" of given topological spaces.

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The notation and terminology used here have been introduced in the following papers: [6], [4], [5], [8], [1], [2], [3], [10], [11], [12], [7], [9], and [13].

1. Open Functions

We adopt the following rules: n, m are elements of $\mathbb{N}$, T is a non empty topological space, and M, M_1, M_2 are non empty metric spaces.

The following propositions are true:

(1) Let A, B, S, T be topological spaces, f be a function from A into S, and g be a function from B into T. Suppose that

(i) the topological structure of $A =$ the topological structure of B,

(ii) the topological structure of $S =$ the topological structure of T,

(iii) $f = g$, and

(iv) f is open.

Then g is open.

(2) Let P be a subset of $\mathcal{E}^m_{\mathrm{T}}$. Then P is open if and only if for every point p of $\mathcal{E}^m_{\mathrm{T}}$ such that $p \in P$ there exists a positive real number r such that $\mathrm{Ball}(p, r) \subseteq P$.

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(3) Let X, Y be non empty topological spaces and f be a function from X into Y. Then f is open if and only if for every point p of X and for every open subset V of X such that $p \in V$ there exists an open subset W of Y such that $f(p) \in W$ and $W \subseteq f^\circ V$.

(4) Let f be a function from T into M_{top}. Then f is open if and only if for every point p of T and for every open subset V of T and for every point q of M such that $q = f(p)$ and $p \in V$ there exists a positive real number r such that $\mathrm{Ball}(q, r) \subseteq f^\circ V$.

(5) Let f be a function from M_{top} into T. Then f is open if and only if for every point p of M and for every positive real number r there exists an open subset W of T such that $f(p) \in W$ and $W \subseteq f^\circ \mathrm{Ball}(p, r)$.

(6) Let f be a function from $(M_1)_{\mathrm{top}}$ into $(M_2)_{\mathrm{top}}$. Then f is open if and only if for every point p of M_1 and for every point q of M_2 and for every positive real number r such that $q = f(p)$ there exists a positive real number s such that $\mathrm{Ball}(q, s) \subseteq f^\circ \mathrm{Ball}(p, r)$.

(7) Let f be a function from T into $\mathcal{E}^m_{\mathrm{T}}$. Then f is open if and only if for every point p of T and for every open subset V of T such that $p \in V$ there exists a positive real number r such that $\mathrm{Ball}(f(p), r) \subseteq f^\circ V$.

(8) Let f be a function from $\mathcal{E}^m_{\mathrm{T}}$ into T. Then f is open if and only if for every point p of $\mathcal{E}^m_{\mathrm{T}}$ and for every positive real number r there exists an open subset W of T such that $f(p) \in W$ and $W \subseteq f^\circ \mathrm{Ball}(p, r)$.

(9) Let f be a function from $\mathcal{E}^m_{\mathrm{T}}$ into $\mathcal{E}^n_{\mathrm{T}}$. Then f is open if and only if for every point p of $\mathcal{E}^m_{\mathrm{T}}$ and for every positive real number r there exists a positive real number s such that $\mathrm{Ball}(f(p), s) \subseteq f^\circ \mathrm{Ball}(p, r)$.

(10) Let f be a function from T into $\mathbb{R}^{\mathbf{1}}$. Then f is open if and only if for every point p of T and for every open subset V of T such that $p \in V$ there exists a positive real number r such that $]f(p) - r, f(p) + r[\subseteq f^\circ V$.

(11) Let f be a function from $\mathbb{R}^{\mathbf{1}}$ into T. Then f is open if and only if for every point p of $\mathbb{R}^{\mathbf{1}}$ and for every positive real number r there exists an open subset V of T such that $f(p) \in V$ and $V \subseteq f^\circ]p - r, p + r[$.

(12) Let f be a function from $\mathbb{R}^{\mathbf{1}}$ into $\mathbb{R}^{\mathbf{1}}$. Then f is open if and only if for every point p of $\mathbb{R}^{\mathbf{1}}$ and for every positive real number r there exists a positive real number s such that $]f(p) - s, f(p) + s[\subseteq f^\circ]p - r, p + r[$.

(13) Let f be a function from $\mathcal{E}^m_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$. Then f is open if and only if for every point p of $\mathcal{E}^m_{\mathrm{T}}$ and for every positive real number r there exists a positive real number s such that $]f(p) - s, f(p) + s[\subseteq f^\circ \mathrm{Ball}(p, r)$.

(14) Let f be a function from $\mathbb{R}^{\mathbf{1}}$ into $\mathcal{E}^m_{\mathrm{T}}$. Then f is open if and only if for every point p of $\mathbb{R}^{\mathbf{1}}$ and for every positive real number r there exists a positive real number s such that $\mathrm{Ball}(f(p), s) \subseteq f^\circ]p - r, p + r[$.

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2. Continuous Functions

Next we state a number of propositions:

(15) Let f be a function from T into M_{top}. Then f is continuous if and only if for every point p of T and for every point q of M and for every positive real number r such that $q = f(p)$ there exists an open subset W of T such that $p \in W$ and $f^\circ W \subseteq \mathrm{Ball}(q, r)$.

(16) Let f be a function from M_{top} into T. Then f is continuous if and only if for every point p of M and for every open subset V of T such that $f(p) \in V$ there exists a positive real number s such that $f^\circ \mathrm{Ball}(p, s) \subseteq V$.

(17) Let f be a function from $(M_1)_{\mathrm{top}}$ into $(M_2)_{\mathrm{top}}$. Then f is continuous if and only if for every point p of M_1 and for every point q of M_2 and for every positive real number r such that $q = f(p)$ there exists a positive real number s such that $f^\circ \mathrm{Ball}(p, s) \subseteq \mathrm{Ball}(q, r)$.

(18) Let f be a function from T into $\mathcal{E}^m_{\mathrm{T}}$. Then f is continuous if and only if for every point p of T and for every positive real number r there exists an open subset W of T such that $p \in W$ and $f^\circ W \subseteq \mathrm{Ball}(f(p), r)$.

(19) Let f be a function from $\mathcal{E}^m_{\mathrm{T}}$ into T. Then f is continuous if and only if for every point p of $\mathcal{E}^m_{\mathrm{T}}$ and for every open subset V of T such that $f(p) \in V$ there exists a positive real number s such that $f^\circ \mathrm{Ball}(p, s) \subseteq V$.

(20) Let f be a function from $\mathcal{E}^m_{\mathrm{T}}$ into $\mathcal{E}^n_{\mathrm{T}}$. Then f is continuous if and only if for every point p of $\mathcal{E}^m_{\mathrm{T}}$ and for every positive real number r there exists a positive real number s such that $f^\circ \mathrm{Ball}(p, s) \subseteq \mathrm{Ball}(f(p), r)$.

(21) Let f be a function from T into $\mathbb{R}^{\mathbf{1}}$. Then f is continuous if and only if for every point p of T and for every positive real number r there exists an open subset W of T such that $p \in W$ and $f^\circ W \subseteq]f(p) - r, f(p) + r[$.

(22) Let f be a function from $\mathbb{R}^{\mathbf{1}}$ into T. Then f is continuous if and only if for every point p of $\mathbb{R}^{\mathbf{1}}$ and for every open subset V of T such that $f(p) \in V$ there exists a positive real number s such that $f^\circ]p - s, p + s[\subseteq V$.

(23) Let f be a function from $\mathbb{R}^{\mathbf{1}}$ into $\mathbb{R}^{\mathbf{1}}$. Then f is continuous if and only if for every point p of $\mathbb{R}^{\mathbf{1}}$ and for every positive real number r there exists a positive real number s such that $f^\circ]p - s, p + s[\subseteq]f(p) - r, f(p) + r[$.

(24) Let f be a function from $\mathcal{E}^m_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$. Then f is continuous if and only if for every point p of $\mathcal{E}^m_{\mathrm{T}}$ and for every positive real number r there exists a positive real number s such that $f^\circ \mathrm{Ball}(p, s) \subseteq]f(p) - r, f(p) + r[$.

(25) Let f be a function from $\mathbb{R}^{\mathbf{1}}$ into $\mathcal{E}^m_{\mathrm{T}}$. Then f is continuous if and only if for every point p of $\mathbb{R}^{\mathbf{1}}$ and for every positive real number r there exists a positive real number s such that $f^\circ]p - s, p + s[\subseteq \mathrm{Ball}(f(p), r)$.

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References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[3] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(**4**):481–485, 1991.
[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[6] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.
[7] Agata Darmochwał and Yatsuka Nakamura. Metric spaces as topological spaces – fundamental concepts. *Formalized Mathematics*, 2(**4**):605–608, 1991.
[8] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(**3**):607–610, 1990.
[9] Artur Korniłowicz and Yasunari Shidama. Intersections of intervals and balls in $\mathcal{E}^n_{\mathrm{T}}$. *Formalized Mathematics*, 12(**3**):301–306, 2004.
[10] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[11] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.
[12] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.
[13] Mariusz Żynel and Adam Guzowski. T_0 topological spaces. *Formalized Mathematics*, 5(**1**):75–77, 1996.

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