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Integrability Formulas. Part III

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Summary. In this article, we give several differentiation and integrability formulas of composite trigonometric function.

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The papers [9], [10], [15], [2], [3], [1], [6], [11], [4], [16], [7], [8], [5], [17], [13], [14], and [12] provide the terminology and notation for this paper.

1. Differentiation Formulas

For simplicity, we adopt the following convention: a, x denote real numbers, n denotes a natural number, A denotes a closed-interval subset of $\mathbb{R}$, f, f_1 denote partial functions from $\mathbb{R}$ to $\mathbb{R}$, and Z denotes an open subset of $\mathbb{R}$.

One can prove the following propositions:

(1) Suppose $Z \subseteq \mathrm{dom}((\text{the function sec}) \cdot \frac{1}{\mathrm{id}_Z})$. Then

(i) $-(\text{the function sec}) \cdot \frac{1}{\mathrm{id}_Z}$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\text{the function sec}) \cdot \frac{1}{\mathrm{id}_Z})'_{\restriction Z}(x) = \frac{(\text{the function sin})(\frac{1}{x})}{x^{\mathbf{2}} \cdot (\text{the function cos})(\frac{1}{x})^{\mathbf{2}}}$.

(2) Suppose $Z \subseteq \mathrm{dom}((\text{the function cosec}) \cdot (\text{the function exp}))$. Then

(i) $-(\text{the function cosec}) \cdot (\text{the function exp})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\text{the function cosec}) \cdot (\text{the function exp}))'_{\restriction Z}(x) = \frac{(\text{the function exp})(x) \cdot (\text{the function cos})((\text{the function exp})(x))}{(\text{the function sin})((\text{the function exp})(x))^{\mathbf{2}}}$.

(3) Suppose $Z \subseteq \mathrm{dom}((\text{the function cosec}) \cdot (\text{the function ln}))$. Then

(i) $-(\text{the function cosec}) \cdot (\text{the function ln})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\text{the function cosec}) \cdot (\text{the function ln}))'_{\restriction Z}(x) = \frac{(\text{the function cos})((\text{the function ln})(x))}{x \cdot (\text{the function sin})((\text{the function ln})(x))^{\mathbf{2}}}$.

(4) Suppose $Z \subseteq \text{dom}((\text{the function exp}) \cdot (\text{the function cosec}))$. Then

(i) $-(\text{the function exp}) \cdot (\text{the function cosec})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\text{the function exp}) \cdot (\text{the function cosec}))'_{\restriction Z}(x) = \frac{(\text{the function exp})((\text{the function cosec})(x)) \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)^{\mathbf{2}}}$.

(5) Suppose $Z \subseteq \text{dom}((\text{the function ln}) \cdot (\text{the function cosec}))$. Then

(i) $-(\text{the function ln}) \cdot (\text{the function cosec})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\text{the function ln}) \cdot (\text{the function cosec}))'_{\restriction Z}(x) = (\text{the function cot})(x)$.

(6) Suppose $Z \subseteq \text{dom}((\square^n) \cdot \text{the function cosec})$ and $1 \leq n$. Then

(i) $-(\square^n) \cdot$ the function cosec is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\square^n) \cdot \text{the function cosec})'_{\restriction Z}(x) = \frac{n \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)^{n+1}}$.

(7) Suppose $Z \subseteq \text{dom}(\frac{1}{\text{id}_Z} \text{ the function sec})$. Then

(i) $-\frac{1}{\text{id}_Z}$ the function sec is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-\frac{1}{\text{id}_Z} \text{ the function sec})'_{\restriction Z}(x) = \frac{\frac{1}{(\text{the function cos})(x)}}{x^{\mathbf{2}}} - \frac{\frac{(\text{the function sin})(x)}{x}}{(\text{the function cos})(x)^{\mathbf{2}}}$.

(8) Suppose $Z \subseteq \text{dom}(\frac{1}{\text{id}_Z} \text{ the function cosec})$. Then

(i) $-\frac{1}{\text{id}_Z}$ the function cosec is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-\frac{1}{\text{id}_Z} \text{ the function cosec})'_{\restriction Z}(x) = \frac{\frac{1}{(\text{the function sin})(x)}}{x^{\mathbf{2}}} + \frac{\frac{(\text{the function cos})(x)}{x}}{(\text{the function sin})(x)^{\mathbf{2}}}$.

(9) Suppose $Z \subseteq \text{dom}((\text{the function cosec}) \cdot (\text{the function sin}))$. Then

(i) $-(\text{the function cosec}) \cdot (\text{the function sin})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\text{the function cosec}) \cdot (\text{the function sin}))'_{\restriction Z}(x) = \frac{(\text{the function cos})(x) \cdot (\text{the function cos})((\text{the function sin})(x))}{(\text{the function sin})((\text{the function sin})(x))^{\mathbf{2}}}$.

(10) Suppose $Z \subseteq \text{dom}((\text{the function sec}) \cdot (\text{the function cot}))$. Then

(i) $-(\text{the function sec}) \cdot (\text{the function cot})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\text{the function sec}) \cdot (\text{the function cot}))'_{\restriction Z}(x) = \frac{\frac{(\text{the function sin})((\text{the function cot})(x))}{(\text{the function sin})(x)^{\mathbf{2}}}}{(\text{the function cos})((\text{the function cot})(x))^{\mathbf{2}}}$.

(11) Suppose $Z \subseteq \text{dom}((\text{the function cosec}) \cdot (\text{the function tan}))$. Then

(i) $-(\text{the function cosec}) \cdot (\text{the function tan})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\text{the function cosec}) \cdot (\text{the function tan}))'_{\restriction Z}(x) = \frac{\frac{(\text{the function cos})((\text{the function tan})(x))}{(\text{the function cos})(x)^{\mathbf{2}}}}{(\text{the function sin})((\text{the function tan})(x))^{\mathbf{2}}}$.

(12) Suppose $Z \subseteq \text{dom}((\text{the function cot}) \ (\text{the function sec}))$. Then

(i) $-(\text{the function cot}) \, (\text{the function sec})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\text{the function cot}) \, (\text{the function sec))$'_{\restriction Z}(x) = \frac{\frac{1}{(\text{the function sin})(x)^{2}}}{(\text{the function cos})(x)} - \frac{(\text{the function cot})(x)\cdot(\text{the function sin})(x)}{(\text{the function cos})(x)^{2}}$.

(13) Suppose $Z \subseteq \mathrm{dom}((\text{the function cot})\ (\text{the function cosec}))$. Then

(i) $-(\text{the function cot})\,(\text{the function cosec})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\text{the function cot})\,(\text{the function}$ cosec))$'_{\restriction Z}(x) = \frac{\frac{1}{(\text{the function sin})(x)^{2}}}{(\text{the function sin})(x)} + \frac{(\text{the function cot})(x)\cdot(\text{the function cos})(x)}{(\text{the function sin})(x)^{2}}$.

(14) Suppose $Z \subseteq \mathrm{dom}((\text{the function cos})\ (\text{the function cot}))$. Then

(i) $-(\text{the function cos})\,(\text{the function cot})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\text{the function cos})\,(\text{the function}$ cot))$'_{\restriction Z}(x) = (\text{the function cos})(x) + \frac{(\text{the function cos})(x)}{(\text{the function sin})(x)^{2}}$.

2. Integrability Formulas

We now state a number of propositions:

(15) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{(\text{the function sin})(\frac{1}{x})}{x^{2}\cdot(\text{the function cos})(\frac{1}{x})^{2}}$,

(iii) $Z \subseteq \mathrm{dom}((\text{the function sec}) \cdot \frac{1}{\mathrm{id}_Z})$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = (-(\text{the function sec}) \cdot \frac{1}{\mathrm{id}_Z})(\sup A) - (-(\text{the function}$ sec) $\cdot \frac{1}{\mathrm{id}_Z})(\inf A)$.

(16) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{(\text{the function cos})(\frac{1}{x})}{x^{2}\cdot(\text{the function sin})(\frac{1}{x})^{2}}$,

(iii) $Z \subseteq \mathrm{dom}((\text{the function cosec}) \cdot \frac{1}{\mathrm{id}_Z})$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = ((\text{the function cosec}) \cdot \frac{1}{\mathrm{id}_Z})(\sup A) - ((\text{the function}$ cosec) $\cdot \frac{1}{\mathrm{id}_Z})(\inf A)$.

(17) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{(\text{the function exp})(x)\cdot(\text{the function sin})((\text{the function exp})(x))}{(\text{the function cos})((\text{the function exp})(x))^{2}}$,

(iii) $Z \subseteq \mathrm{dom}((\text{the function sec}) \cdot (\text{the function exp}))$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx =$ ((the function sec) $\cdot$(the function exp))(sup A)$-$((the function sec) $\cdot$(the function exp))(inf A).

(18) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\text{(the function exp)}(x)\cdot\text{(the function cos)}(\text{(the function exp)}(x))}{\text{(the function sin)}(\text{(the function exp)}(x))^{\mathbf{2}}}$,

(iii) $Z \subseteq$ dom((the function cosec) $\cdot$(the function exp)),

(iv) $Z = \operatorname{dom} f$, and

(v) $f\restriction A$ is continuous.

Then $\int\limits_A f(x)dx = (-$(the function cosec) $\cdot$ (the function exp))(sup A) $-$ $(-$(the function cosec) $\cdot$ (the function exp))(inf A).

(19) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\text{(the function sin)}(\text{(the function ln)}(x))}{x\cdot\text{(the function cos)}(\text{(the function ln)}(x))^{\mathbf{2}}}$,

(iii) $Z \subseteq$ dom((the function sec) $\cdot$(the function ln)),

(iv) $Z = \operatorname{dom} f$, and

(v) $f\restriction A$ is continuous.

Then $\int\limits_A f(x)dx =$ ((the function sec) $\cdot$(the function ln))(sup A) $-$ ((the function sec) $\cdot$(the function ln))(inf A).

(20) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\text{(the function cos)}(\text{(the function ln)}(x))}{x\cdot\text{(the function sin)}(\text{(the function ln)}(x))^{\mathbf{2}}}$,

(iii) $Z \subseteq$ dom((the function cosec) $\cdot$(the function ln)),

(iv) $Z = \operatorname{dom} f$, and

(v) $f\restriction A$ is continuous.

Then $\int\limits_A f(x)dx = (-$(the function cosec) $\cdot$ (the function ln))(sup A) $-$ $(-$(the function cosec) $\cdot$ (the function ln))(inf A).

(21) Suppose that

(i) $A \subseteq Z$,

(ii) $f =$ ((the function exp) $\cdot$(the function sec)) $\frac{\text{the function sin}}{\text{(the function cos)}^{\mathbf{2}}}$,

(iii) $Z = \operatorname{dom} f$, and

(iv) $f\restriction A$ is continuous.

Then $\int\limits_A f(x)dx = ((\text{the function exp}) \cdot (\text{the function sec}))(\sup A) - ((\text{the function exp}) \cdot (\text{the function sec}))(\inf A)$.

(22) Suppose that

(i) $A \subseteq Z$,

(ii) $f = ((\text{the function exp}) \cdot (\text{the function cosec}))\ \frac{\text{the function cos}}{(\text{the function sin})^{\mathbf{2}}}$,

(iii) $Z = \operatorname{dom} f$, and

(iv) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = (-(\text{the function exp}) \cdot (\text{the function cosec}))(\sup A) - (-(\text{the function exp}) \cdot (\text{the function cosec}))(\inf A)$.

(23) Suppose that

(i) $A \subseteq Z$,

(ii) $Z \subseteq \operatorname{dom}((\text{the function ln}) \cdot (\text{the function sec}))$,

(iii) $Z = \operatorname{dom}(\text{the function tan})$, and

(iv) $(\text{the function tan}){\restriction}A$ is continuous.

Then $\int\limits_A (\text{the function tan})(x)dx = ((\text{the function ln}) \cdot (\text{the function sec}))(\sup A) - ((\text{the function ln}) \cdot (\text{the function sec}))(\inf A)$.

(24) Suppose that

(i) $A \subseteq Z$,

(ii) $Z \subseteq \operatorname{dom}((\text{the function ln}) \cdot (\text{the function cosec}))$,

(iii) $Z = \operatorname{dom}(\text{the function cot})$, and

(iv) $(-\text{the function cot}){\restriction}A$ is continuous.

Then $\int\limits_A (-\text{the function cot})(x)dx = ((\text{the function ln}) \cdot (\text{the function cosec}))(\sup A) - ((\text{the function ln}) \cdot (\text{the function cosec}))(\inf A)$.

(25) Suppose that

(i) $A \subseteq Z$,

(ii) $Z \subseteq \operatorname{dom}((\text{the function ln}) \cdot (\text{the function cosec}))$,

(iii) $Z = \operatorname{dom}(\text{the function cot})$, and

(iv) $(\text{the function cot}){\restriction}A$ is continuous.

Then $\int\limits_A (\text{the function cot})(x)dx = (-(\text{the function ln}) \cdot (\text{the function cosec}))(\sup A) - (-(\text{the function ln}) \cdot (\text{the function cosec}))(\inf A)$.

(26) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{n \cdot (\text{the function sin})(x)}{(\text{the function cos})(x)^{n+1}}$,

(iii) $Z \subseteq \operatorname{dom}((\square^n) \cdot \text{the function sec})$,

(iv) $1 \leq n$,

(v) $Z = \mathrm{dom}\, f$, and

(vi) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = ((\square^n) \cdot \text{the function sec})(\sup A) - ((\square^n) \cdot \text{the function sec})(\inf A)$.

(27) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{n\cdot(\text{the function cos})(x)}{(\text{the function sin})(x)^{n+1}}$,

(iii) $Z \subseteq \mathrm{dom}((\square^n) \cdot \text{the function cosec})$,

(iv) $1 \leq n$,

(v) $Z = \mathrm{dom}\, f$, and

(vi) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = (-(\square^n) \cdot \text{the function cosec})(\sup A) - (-(\square^n) \cdot \text{the function cosec})(\inf A)$.

(28) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{(\text{the function exp})(x)}{(\text{the function cos})(x)} + \frac{(\text{the function exp})(x)\cdot(\text{the function sin})(x)}{(\text{the function cos})(x)^{\mathbf{2}}}$,

(iii) $Z \subseteq \mathrm{dom}((\text{the function exp})\ (\text{the function sec}))$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = ((\text{the function exp})\ (\text{the function sec}))(\sup A) - ((\text{the function exp})\ (\text{the function sec}))(\inf A)$.

(29) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{(\text{the function exp})(x)}{(\text{the function sin})(x)} - \frac{(\text{the function exp})(x)\cdot(\text{the function cos})(x)}{(\text{the function sin})(x)^{\mathbf{2}}}$,

(iii) $Z \subseteq \mathrm{dom}((\text{the function exp})\ (\text{the function cosec}))$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = ((\text{the function exp})\ (\text{the function cosec}))(\sup A) - ((\text{the function exp})\ (\text{the function cosec}))(\inf A)$.

(30) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds
$f(x) = \frac{(\text{the function sin})(a\cdot x) - (\text{the function cos})(a\cdot x)^{\mathbf{2}}}{(\text{the function cos})(a\cdot x)^{\mathbf{2}}}$,

(iii) $Z \subseteq \operatorname{dom}(\frac{1}{a}$ ((the function sec) $\cdot f_1) - \mathrm{id}_Z)$,
(iv) for every x such that $x \in Z$ holds $f_1(x) = a \cdot x$ and $a \neq 0$,
(v) $Z = \operatorname{dom} f$, and
(vi) $f \restriction A$ is continuous.
Then $\int\limits_A f(x)dx = (\frac{1}{a}$ ((the function sec) $\cdot f_1) - \mathrm{id}_Z)(\sup A) - (\frac{1}{a}$ ((the function sec) $\cdot f_1) - \mathrm{id}_Z)(\inf A)$.

(31) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = \frac{(\text{the function cos})(a \cdot x) - (\text{the function sin})(a \cdot x)^{\mathbf{2}}}{(\text{the function sin})(a \cdot x)^{\mathbf{2}}}$,
(iii) $Z \subseteq \operatorname{dom}((-\frac{1}{a})$ ((the function cosec) $\cdot f_1) - \mathrm{id}_Z)$,
(iv) for every x such that $x \in Z$ holds $f_1(x) = a \cdot x$ and $a \neq 0$,
(v) $Z = \operatorname{dom} f$, and
(vi) $f \restriction A$ is continuous.
Then $\int\limits_A f(x)dx = ((-\frac{1}{a})$ ((the function cosec) $\cdot f_1) - \mathrm{id}_Z)(\sup A) - ((-\frac{1}{a})$ ((the function cosec) $\cdot f_1) - \mathrm{id}_Z)(\inf A)$.

(32) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\frac{1}{(\text{the function cos})(x)}}{x} + \frac{(\text{the function ln})(x) \cdot (\text{the function sin})(x)}{(\text{the function cos})(x)^{\mathbf{2}}}$,
(iii) $Z \subseteq \operatorname{dom}$((the function ln) (the function sec)),
(iv) $Z = \operatorname{dom} f$, and
(v) $f \restriction A$ is continuous.
Then $\int\limits_A f(x)dx =$ ((the function ln) (the function sec))$(\sup A)$ − ((the function ln) (the function sec))$(\inf A)$.

(33) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\frac{1}{(\text{the function sin})(x)}}{x} - \frac{(\text{the function ln})(x) \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)^{\mathbf{2}}}$,
(iii) $Z \subseteq \operatorname{dom}$((the function ln) (the function cosec)),
(iv) $Z = \operatorname{dom} f$, and
(v) $f \restriction A$ is continuous.
Then $\int\limits_A f(x)dx =$ ((the function ln) (the function cosec))$(\sup A)$ − ((the function ln) (the function cosec))$(\inf A)$.

(34) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\frac{1}{(\text{the function }\cos)(x)}}{x^2} - \frac{\frac{(\text{the function }\sin)(x)}{x}}{(\text{the function }\cos)(x)^2}$,

(iii) $Z \subseteq \operatorname{dom}(\frac{1}{\mathrm{id}_Z}$ the function sec),

(iv) $Z = \operatorname{dom} f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = (-\frac{1}{\mathrm{id}_Z}$ the function sec)$(\sup A) - (-\frac{1}{\mathrm{id}_Z}$ the function sec)$(\inf A)$.

(35) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\frac{1}{(\text{the function }\sin)(x)}}{x^2} + \frac{\frac{(\text{the function }\cos)(x)}{x}}{(\text{the function }\sin)(x)^2}$,

(iii) $Z \subseteq \operatorname{dom}(\frac{1}{\mathrm{id}_Z}$ the function cosec),

(iv) $Z = \operatorname{dom} f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = (-\frac{1}{\mathrm{id}_Z}$ the function cosec)$(\sup A) - (-\frac{1}{\mathrm{id}_Z}$ the function cosec)$(\inf A)$.

(36) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{(\text{the function }\cos)(x) \cdot (\text{the function }\sin)((\text{the function }\sin)(x))}{(\text{the function }\cos)((\text{the function }\sin)(x))^2}$,

(iii) $Z \subseteq \operatorname{dom}((\text{the function sec}) \cdot (\text{the function sin}))$,

(iv) $Z = \operatorname{dom} f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = ((\text{the function sec}) \cdot (\text{the function sin}))(\sup A) - ((\text{the function sec}) \cdot (\text{the function sin}))(\inf A)$.

(37) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{(\text{the function }\sin)(x) \cdot (\text{the function }\sin)((\text{the function }\cos)(x))}{(\text{the function }\cos)((\text{the function }\cos)(x))^2}$,

(iii) $Z \subseteq \operatorname{dom}((\text{the function sec}) \cdot (\text{the function cos}))$,

(iv) $Z = \operatorname{dom} f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = (-(\text{the function sec}) \cdot (\text{the function cos}))(\sup A) - (-(\text{the function sec}) \cdot (\text{the function cos}))(\inf A)$.

(38) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{(\text{the function cos})(x) \cdot (\text{the function cos})((\text{the function sin})(x))}{(\text{the function sin})((\text{the function sin})(x))^{2}}$,

(iii) $Z \subseteq \mathrm{dom}((\text{the function cosec}) \cdot (\text{the function sin}))$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f \restriction A$ is continuous.

Then $\int\limits_{A} f(x)dx = (-(\text{the function cosec}) \cdot (\text{the function sin}))(\sup A) - (-(\text{the function cosec}) \cdot (\text{the function sin}))(\inf A)$.

(39) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{(\text{the function sin})(x) \cdot (\text{the function cos})((\text{the function cos})(x))}{(\text{the function sin})((\text{the function cos})(x))^{2}}$,

(iii) $Z \subseteq \mathrm{dom}((\text{the function cosec}) \cdot (\text{the function cos}))$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f \restriction A$ is continuous.

Then $\int\limits_{A} f(x)dx = ((\text{the function cosec}) \cdot (\text{the function cos}))(\sup A) - ((\text{the function cosec}) \cdot (\text{the function cos}))(\inf A)$.

(40) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\frac{(\text{the function sin})((\text{the function tan})(x))}{(\text{the function cos})(x)^{2}}}{(\text{the function cos})((\text{the function tan})(x))^{2}}$,

(iii) $Z \subseteq \mathrm{dom}((\text{the function sec}) \cdot (\text{the function tan}))$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f \restriction A$ is continuous.

Then $\int\limits_{A} f(x)dx = ((\text{the function sec}) \cdot (\text{the function tan}))(\sup A) - ((\text{the function sec}) \cdot (\text{the function tan}))(\inf A)$.

(41) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\frac{(\text{the function sin})((\text{the function cot})(x))}{(\text{the function sin})(x)^{2}}}{(\text{the function cos})((\text{the function cot})(x))^{2}}$,

(iii) $Z \subseteq \mathrm{dom}((\text{the function sec}) \cdot (\text{the function cot}))$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f \restriction A$ is continuous.

Then $\int\limits_{A} f(x)dx = (-(\text{the function sec}) \cdot (\text{the function cot}))(\sup A) - (-(\text{the function sec}) \cdot (\text{the function cot}))(\inf A)$.

(42) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\frac{(\text{the function cos})((\text{the function tan})(x))}{(\text{the function cos})(x)^{\mathbf{2}}}}{(\text{the function sin})((\text{the function tan})(x))^{\mathbf{2}}}$,

(iii) $Z \subseteq \mathrm{dom}((\text{the function cosec}) \cdot (\text{the function tan}))$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f \restriction A$ is continuous.

Then $\int\limits_A f(x)dx = (-(\text{the function cosec}) \cdot (\text{the function tan}))(\sup A) - (-(\text{the function cosec}) \cdot (\text{the function tan}))(\inf A)$.

(43) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\frac{(\text{the function cos})((\text{the function cot})(x))}{(\text{the function sin})(x)^{\mathbf{2}}}}{(\text{the function sin})((\text{the function cot})(x))^{\mathbf{2}}}$,

(iii) $Z \subseteq \mathrm{dom}((\text{the function cosec}) \cdot (\text{the function cot}))$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f \restriction A$ is continuous.

Then $\int\limits_A f(x)dx = ((\text{the function cosec}) \cdot (\text{the function cot}))(\sup A) - ((\text{the function cosec}) \cdot (\text{the function cot}))(\inf A)$.

(44) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\frac{1}{(\text{the function cos})(x)^{\mathbf{2}}}}{(\text{the function cos})(x)} + \frac{(\text{the function tan})(x) \cdot (\text{the function sin})(x)}{(\text{the function cos})(x)^{\mathbf{2}}}$,

(iii) $Z \subseteq \mathrm{dom}((\text{the function tan})\ (\text{the function sec}))$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f \restriction A$ is continuous.

Then $\int\limits_A f(x)dx = ((\text{the function tan})\ (\text{the function sec}))(\sup A) - ((\text{the function tan})\ (\text{the function sec}))(\inf A)$.

(45) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\frac{1}{(\text{the function sin})(x)^{\mathbf{2}}}}{(\text{the function cos})(x)} - \frac{(\text{the function cot})(x) \cdot (\text{the function sin})(x)}{(\text{the function cos})(x)^{\mathbf{2}}}$,

(iii) $Z \subseteq \mathrm{dom}((\text{the function cot})\ (\text{the function sec}))$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f \restriction A$ is continuous.

Then $\int\limits_A f(x)dx = (-(\text{the function cot})\ (\text{the function sec}))(\sup A) - (-(\text{the$

function cot) (the function sec))(inf A).

(46) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\frac{1}{(\text{the function cos})(x)^{\mathbf{2}}}}{(\text{the function sin})(x)} - \frac{(\text{the function tan})(x) \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)^{\mathbf{2}}}$,

(iii) $Z \subseteq \mathrm{dom}((\text{the function tan})\ (\text{the function cosec}))$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = ((\text{the function tan})\ (\text{the function cosec}))(\sup A) - ((\text{the function tan})\ (\text{the function cosec}))(\inf A)$.

(47) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\frac{1}{(\text{the function sin})(x)^{\mathbf{2}}}}{(\text{the function sin})(x)} + \frac{(\text{the function cot})(x) \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)^{\mathbf{2}}}$,

(iii) $Z \subseteq \mathrm{dom}((\text{the function cot})\ (\text{the function cosec}))$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = (-(\text{the function cot})\,(\text{the function cosec}))(\sup A) - (-(\text{the function cot})\,(\text{the function cosec}))(\inf A)$.

(48) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds
$f(x) = \frac{1}{(\text{the function cos})((\text{the function cot})(x))^{\mathbf{2}}} \cdot \frac{1}{(\text{the function sin})(x)^{\mathbf{2}}}$,

(iii) $Z \subseteq \mathrm{dom}((\text{the function tan}) \cdot (\text{the function cot}))$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = (-(\text{the function tan}) \cdot (\text{the function cot}))(\sup A) - (-(\text{the function tan}) \cdot (\text{the function cot}))(\inf A)$.

(49) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds
$f(x) = \frac{1}{(\text{the function cos})((\text{the function tan})(x))^{\mathbf{2}}} \cdot \frac{1}{(\text{the function cos})(x)^{\mathbf{2}}}$,

(iii) $Z \subseteq \mathrm{dom}((\text{the function tan}) \cdot (\text{the function tan}))$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = ((\text{the function tan}) \cdot (\text{the function tan}))(\sup A) - ((\text{the function tan}) \cdot (\text{the function tan}))(\inf A)$.

(50) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{1}{(\text{the function sin})((\text{the function cot})(x))^{\mathbf{2}}} \cdot \frac{1}{(\text{the function sin})(x)^{\mathbf{2}}}$,

(iii) $Z \subseteq \operatorname{dom}((\text{the function cot}) \cdot (\text{the function cot}))$,

(iv) $Z = \operatorname{dom} f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = ((\text{the function cot}) \cdot (\text{the function cot}))(\sup A) - ((\text{the function cot}) \cdot (\text{the function cot}))(\inf A)$.

(51) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{1}{(\text{the function sin})((\text{the function tan})(x))^{\mathbf{2}}} \cdot \frac{1}{(\text{the function cos})(x)^{\mathbf{2}}}$,

(iii) $Z \subseteq \operatorname{dom}((\text{the function cot}) \cdot (\text{the function tan}))$,

(iv) $Z = \operatorname{dom} f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = (-(\text{the function cot}) \cdot (\text{the function tan}))(\sup A) - (-(\text{the function cot}) \cdot (\text{the function tan}))(\inf A)$.

(52) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{1}{(\text{the function cos})(x)^{\mathbf{2}}} + \frac{1}{(\text{the function sin})(x)^{\mathbf{2}}}$,

(iii) $Z \subseteq \operatorname{dom}((\text{the function tan}) - (\text{the function cot}))$,

(iv) $Z = \operatorname{dom} f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = ((\text{the function tan}) - (\text{the function cot}))(\sup A) - ((\text{the function tan}) - (\text{the function cot}))(\inf A)$.

(53) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{1}{(\text{the function cos})(x)^{\mathbf{2}}} - \frac{1}{(\text{the function sin})(x)^{\mathbf{2}}}$,

(iii) $Z \subseteq \operatorname{dom}((\text{the function tan}) + (\text{the function cot}))$,

(iv) $Z = \operatorname{dom} f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = (($the function tan$)+($the function cot$))(\sup A) - (($the function tan$)+($the function cot$))(\inf A)$.

(54) Suppose that
 (i) $A \subseteq Z$,
 (ii) for every x such that $x \in Z$ holds $f(x) =$ (the function cos)((the function sin)(x)) $\cdot$ (the function cos)(x),
 (iii) $Z = \operatorname{dom} f$, and
 (iv) $f\restriction A$ is continuous.

Then $\int\limits_A f(x)dx = (($the function sin$) \cdot ($the function sin$))(\sup A) - (($the function sin$) \cdot ($the function sin$))(\inf A)$.

(55) Suppose that
 (i) $A \subseteq Z$,
 (ii) for every x such that $x \in Z$ holds $f(x) =$ (the function cos)((the function cos)(x)) $\cdot$ (the function sin)(x),
 (iii) $Z = \operatorname{dom} f$, and
 (iv) $f\restriction A$ is continuous.

Then $\int\limits_A f(x)dx = (-($the function sin$) \cdot ($the function cos$))(\sup A) - (-($the function sin$) \cdot ($the function cos$))(\inf A)$.

(56) Suppose that
 (i) $A \subseteq Z$,
 (ii) for every x such that $x \in Z$ holds $f(x) =$ (the function sin)((the function sin)(x)) $\cdot$ (the function cos)(x),
 (iii) $Z = \operatorname{dom} f$, and
 (iv) $f\restriction A$ is continuous.

Then $\int\limits_A f(x)dx = (-($the function cos$) \cdot ($the function sin$))(\sup A) - (-($the function cos$) \cdot ($the function sin$))(\inf A)$.

(57) Suppose that
 (i) $A \subseteq Z$,
 (ii) for every x such that $x \in Z$ holds $f(x) =$ (the function sin)((the function cos)(x)) $\cdot$ (the function sin)(x),
 (iii) $Z = \operatorname{dom} f$, and
 (iv) $f\restriction A$ is continuous.

Then $\int\limits_A f(x)dx = (($the function cos$) \cdot ($the function cos$))(\sup A) - (($the function cos$) \cdot ($the function cos$))(\inf A)$.

(58) Suppose that
 (i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) =$ (the function cos)$(x) + \frac{(\text{the function cos})(x)}{(\text{the function sin})(x)^{\mathbf{2}}}$,
(iii) $Z \subseteq \operatorname{dom}$((the function cos) (the function cot)),
(iv) $Z = \operatorname{dom} f$, and
(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = (-$(the function cos) (the function cot))$(\sup A) - (-$(the function cos) (the function cot))$(\inf A)$.

(59) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) =$ (the function sin)$(x) + \frac{(\text{the function sin})(x)}{(\text{the function cos})(x)^{\mathbf{2}}}$,
(iii) $Z \subseteq \operatorname{dom}$((the function sin) (the function tan)),
(iv) $Z = \operatorname{dom} f$, and
(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx =$ ((the function sin) (the function tan))$(\sup A) -$ ((the function sin) (the function tan))$(\inf A)$.

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