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Abstract Simplicial Complexes

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Summary. In this article we define the notion of abstract simplicial complexes and operations on them. We introduce the following basic notions: simplex, face, vertex, degree, skeleton, subdivision and substructure, and prove some of their properties.

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The articles [2], [5], [6], [10], [8], [14], [1], [7], [3], [4], [11], [13], [16], [12], [15], and [9] provide the notation and terminology for this paper.

1. Preliminaries

For simplicity, we adopt the following convention: x, y, X, Y, Z are sets, D is a non empty set, n, k are natural numbers, and i, i_1, i_2 are integers.

Let us consider X. We introduce X has empty element as an antonym of X has non empty elements.

Note that there exists a set which is empty and finite-membered and every set which is empty is also finite-membered. Let X be a finite set. Note that $\{X\}$ is finite-membered and 2^X is finite-membered. Let Y be a finite set. Observe that $\{X, Y\}$ is finite-membered.

Let X be a finite-membered set. Observe that every subset of X is finite-membered. Let Y be a finite-membered set. One can check that $X \cup Y$ is finite-membered.

Let X be a finite finite-membered set. Note that $\bigcup X$ is finite.

One can verify the following observations:

* every set which is empty is also subset-closed,

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* every set which has empty element is also non empty,
* every set which is non empty and subset-closed has also empty element, and
* there exists a set which has empty element.

Let us consider X. Observe that $\mathrm{SubFin}(X)$ is finite-membered and there exists a family of subsets of X which is subset-closed, finite, and finite-membered.

Let X be a subset-closed set. One can check that $\mathrm{SubFin}(X)$ is subset-closed.

Next we state the proposition

(1) Y is subset-closed iff for every X such that $X \in Y$ holds $2^X \subseteq Y$.

Let A, B be subset-closed sets. Note that $A \cup B$ is subset-closed and $A \cap B$ is subset-closed.

Let us consider X. The subset-closure of X yields a subset-closed set and is defined by the conditions (Def. 1).

(Def. 1)(i) $X \subseteq$ the subset-closure of X, and

(ii) for every Y such that $X \subseteq Y$ and Y is subset-closed holds the subset-closure of $X \subseteq Y$.

The following proposition is true

(2) $x \in$ the subset-closure of X iff there exists y such that $x \subseteq y$ and $y \in X$.

Let us consider X and let F be a family of subsets of X. Then the subset-closure of F is a subset-closed family of subsets of X.

Observe that the subset-closure of $\emptyset$ is empty. Let X be a non empty set. Note that the subset-closure of X is non empty.

Let X be a set with a non-empty element. One can check that the subset-closure of X has a non-empty element.

Let X be a finite-membered set. Note that the subset-closure of X is finite-membered.

The following propositions are true:

(3) If $X \subseteq Y$ and Y is subset-closed, then the subset-closure of $X \subseteq Y$.

(4) The subset-closure of $\{X\} = 2^X$.

(5) The subset-closure of $X \cup Y =$ (the subset-closure of X) $\cup$ (the subset-closure of Y).

(6) X is finer than Y iff the subset-closure of $X \subseteq$ the subset-closure of Y.

(7) If X is subset-closed, then the subset-closure of $X = X$.

(8) If the subset-closure of $X \subseteq X$, then X is subset-closed.

Let us consider Y, X and let n be a set. The subsets of X and Y with cardinality limited by n yields a family of subsets of Y and is defined by the condition (Def. 2).

(Def. 2) Let A be a subset of Y. Then $A \in$ the subsets of X and Y with cardinality limited by n if and only if $A \in X$ and $\mathrm{Card}\,A \subseteq \mathrm{Card}\,n$.

Let us consider D. One can verify that there exists a family of subsets of D which is finite, subset-closed, and finite-membered and has a non-empty element.

Let us consider Y, X and let n be a finite set. One can check that the subsets of X and Y with cardinality limited by n is finite-membered.

Let us consider Y, let X be a subset-closed set, and let n be a set. Note that the subsets of X and Y with cardinality limited by n is subset-closed.

Let us consider Y, let X be a set with empty element, and let n be a set. One can check that the subsets of X and Y with cardinality limited by n has empty element.

Let us consider D, let X be a subset-closed family of subsets of D with a non-empty element, and let n be a non empty set. Note that the subsets of X and D with cardinality limited by n has a non-empty element.

Let us consider X, let Y be a family of subsets of X, and let n be a set. We introduce the subsets of Y with cardinality limited by n as a synonym of the subsets of Y and X with cardinality limited by n.

Let us observe that every set which is empty is also $\subseteq$-linear and there exists a set which is empty and $\subseteq$-linear.

Let X be a $\subseteq$-linear set. Note that every subset of X is $\subseteq$-linear.

The following propositions are true:

(9) If X is non empty, finite, and $\subseteq$-linear, then $\bigcup X \in X$.

(10) For every finite $\subseteq$-linear set X such that X has non empty elements holds $\operatorname{Card} X \subseteq \operatorname{Card} \bigcup X$.

(11) If X is $\subseteq$-linear and $\bigcup X$ misses x, then $X \cup \{\bigcup X \cup x\}$ is $\subseteq$-linear.

(12) Let X be a non empty set. Then there exists a family Y of subsets of X such that

(i) Y is $\subseteq$-linear and has non empty elements,

(ii) $X \in Y$,

(iii) $\operatorname{Card} X = \operatorname{Card} Y$, and

(iv) for every Z such that $Z \in Y$ and $\operatorname{Card} Z \neq 1$ there exists x such that $x \in Z$ and $Z \setminus \{x\} \in Y$.

(13) Let Y be a family of subsets of X. Suppose Y is finite and $\subseteq$-linear and has non empty elements and $X \in Y$. Then there exists a family Y' of subsets of X such that

(i) $Y \subseteq Y'$,

(ii) Y' is $\subseteq$-linear and has non empty elements,

(iii) $\operatorname{Card} X = \operatorname{Card} Y'$, and

(iv) for every Z such that $Z \in Y'$ and $\operatorname{Card} Z \neq 1$ there exists x such that $x \in Z$ and $Z \setminus \{x\} \in Y'$.

2. Simplicial Complexes

A simplicial complex structure is a topological structure.

In the sequel K denotes a simplicial complex structure.

Let us consider K and let A be a subset of K. We introduce A is simplex-like as a synonym of A is open.

Let us consider K and let S be a family of subsets of K. We introduce S is simplex-like as a synonym of S is open.

Let us consider K. One can check that there exists a family of subsets of K which is empty and simplex-like.

The following proposition is true

(14) For every family S of subsets of K holds S is simplex-like iff $S \subseteq$ the topology of K.

Let us consider K and let v be an element of K. We say that v is vertex-like if and only if:

(Def. 3) There exists a subset S of K such that S is simplex-like and $v \in S$.

Let us consider K. The functor Vertices K yielding a subset of K is defined by:

(Def. 4) For every element v of K holds $v \in$ Vertices K iff v is vertex-like.

Let K be a simplicial complex structure. A vertex of K is an element of Vertices K.

Let K be a simplicial complex structure. We say that K is finite-vertices if and only if:

(Def. 5) Vertices K is finite.

Let us consider K. We say that K is locally-finite if and only if:

(Def. 6) For every vertex v of K holds $\{S \subseteq K\colon S \text{ is simplex-like} \wedge v \in S\}$ is finite.

Let us consider K. We say that K is empty-membered if and only if:

(Def. 7) The topology of K is empty-membered.

We say that K has non empty elements if and only if:

(Def. 8) The topology of K has non empty elements.

Let us consider K. We introduce K has a non-empty element as an antonym of K is empty-membered. We introduce K has empty element as an antonym of K has non empty elements.

Let us consider X. A simplicial complex structure is said to be a simplicial complex structure of X if:

(Def. 9) $\Omega_{\mathrm{it}} \subseteq X$.

Let us consider X and let K_1 be a simplicial complex structure of X. We say that K_1 is total if and only if:

(Def. 10) $\Omega_{(K_1)} = X$.

One can check the following observations:

* every simplicial complex structure which has empty element is also non void,
* every simplicial complex structure which has a non-empty element is also non void,
* every simplicial complex structure which is non void and empty-membered has also empty element,
* every simplicial complex structure which is non void and subset-closed has also empty element,
* every simplicial complex structure which is empty-membered is also subset-closed and finite-vertices,
* every simplicial complex structure which is finite-vertices is also locally-finite and finite-degree, and
* every simplicial complex structure which is locally-finite and subset-closed is also finite-membered.

Let us consider X. Observe that there exists a simplicial complex structure of X which is empty, void, empty-membered, and strict.

Let us consider D. Note that there exists a simplicial complex structure of D which is non empty, non void, total, empty-membered, and strict and there exists a simplicial complex structure of D which is non empty, total, finite-vertices, subset-closed, and strict and has empty element and a non-empty element.

Let us observe that there exists a simplicial complex structure which is non empty, finite-vertices, subset-closed, and strict and has empty element and a non-empty element.

Let K be a simplicial complex structure with a non-empty element. Observe that $\mathrm{Vertices}\, K$ is non empty.

Let K be a finite-vertices simplicial complex structure. Note that every family of subsets of K which is simplex-like is also finite.

Let K be a finite-membered simplicial complex structure. Note that every family of subsets of K which is simplex-like is also finite-membered.

Next we state several propositions:

(15) $\mathrm{Vertices}\, K$ is empty iff K is empty-membered.

(16) $\mathrm{Vertices}\, K = \bigcup(\text{the topology of } K)$.

(17) For every subset S of K such that S is simplex-like holds $S \subseteq \mathrm{Vertices}\, K$.

(18) If K is finite-vertices, then the topology of K is finite.

(19) If the topology of K is finite and K is non finite-vertices, then K is non finite-membered.

(20) If K is subset-closed and the topology of K is finite, then K is finite-vertices.

3. The Simplicial Complex Generated on the Set

Let us consider X and let Y be a family of subsets of X. The complex of Y yielding a strict simplicial complex structure of X is defined as follows:

(Def. 11) The complex of $Y = \langle X, \text{the subset-closure of } Y \rangle$.

Let us consider X and let Y be a family of subsets of X. One can verify that the complex of Y is total and subset-closed.

Let us consider X and let Y be a non empty family of subsets of X. Note that the complex of Y has empty element.

Let us consider X and let Y be a finite-membered family of subsets of X. Note that the complex of Y is finite-membered.

Let us consider X and let Y be a finite finite-membered family of subsets of X. Observe that the complex of Y is finite-vertices.

One can prove the following proposition

(21) If K is subset-closed, then the topological structure of $K =$ the complex of the topology of K.

Let us consider X. A simplicial complex of X is a finite-membered subset-closed simplicial complex structure of X.

Let K be a non void simplicial complex structure. A simplex of K is a simplex-like subset of K.

Let K be a simplicial complex structure with empty element. One can check that every subset of K which is empty is also simplex-like and there exists a simplex of K which is empty.

Let K be a non void finite-membered simplicial complex structure. Note that there exists a simplex of K which is finite.

4. The Degree of Simplicial Complexes

Let us consider K. The functor degree(K) yields an extended real number and is defined as follows:

(Def. 12)(i) For every finite subset S of K such that S is simplex-like holds $\overline{\overline{S}} \leq$ degree$(K) + 1$ and there exists a subset S of K such that S is simplex-like and $\operatorname{Card} S =$ degree$(K) + 1$ if K is non void and finite-degree,

(ii) degree$(K) = -1$ if K is void,

(iii) degree$(K) = +\infty$, otherwise.

Let K be a finite-degree simplicial complex structure. Note that degree$(K)+1$ is natural and degree(K) is integer.

The following propositions are true:

(22) degree$(K) = -1$ iff K is empty-membered.

(23) $-1 \leq$ degree(K).

(24) For every finite subset S of K such that S is simplex-like holds $\overline{\overline{S}} \leq \mathrm{degree}(K) + 1$.

(25) Suppose K is non void or $i \geq -1$. Then $\mathrm{degree}(K) \leq i$ if and only if the following conditions are satisfied:

(i) K is finite-membered, and

(ii) for every finite subset S of K such that S is simplex-like holds $\overline{\overline{S}} \leq i+1$.

(26) For every finite subset A of X holds degree(the complex of $\{A\}$) $= \overline{\overline{A}} - 1$.

5. SUBCOMPLEXES

Let us consider X and let K_1 be a simplicial complex structure of X. A simplicial complex of X is said to be a subsimplicial complex of K_1 if:

(Def. 13) $\Omega_{\mathrm{it}} \subseteq \Omega_{(K_1)}$ and the topology of it $\subseteq$ the topology of K_1.

In the sequel K_1 denotes a simplicial complex structure of X and S_1 denotes a subsimplicial complex of K_1.

Let us consider X, K_1. One can check that there exists a subsimplicial complex of K_1 which is empty, void, and strict.

Let us consider X and let K_1 be a void simplicial complex structure of X. Observe that every subsimplicial complex of K_1 is void.

Let us consider D and let K_2 be a non void subset-closed simplicial complex structure of D. Note that there exists a subsimplicial complex of K_2 which is non void.

Let us consider X and let K_1 be a finite-vertices simplicial complex structure of X. One can check that every subsimplicial complex of K_1 is finite-vertices.

Let us consider X and let K_1 be a finite-degree simplicial complex structure of X. Note that every subsimplicial complex of K_1 is finite-degree.

Next we state several propositions:

(27) Every subsimplicial complex of S_1 is a subsimplicial complex of K_1.

(28) Let A be a subset of K_1 and S be a finite-membered family of subsets of A. Suppose the subset-closure of $S \subseteq$ the topology of K_1. Then the complex of S is a strict subsimplicial complex of K_1.

(29) Let K_1 be a subset-closed simplicial complex structure of X, A be a subset of K_1, and S be a finite-membered family of subsets of A. Suppose $S \subseteq$ the topology of K_1. Then the complex of S is a strict subsimplicial complex of K_1.

(30) Let Y_1, Y_2 be families of subsets of X. Suppose Y_1 is finite-membered and finer than Y_2. Then the complex of Y_1 is a subsimplicial complex of the complex of Y_2.

(31) $\mathrm{Vertices}\, S_1 \subseteq \mathrm{Vertices}\, K_1$.

(32) $\mathrm{degree}(S_1) \leq \mathrm{degree}(K_1)$.

Let us consider X, K_1, S_1. We say that S_1 is maximal if and only if:

(Def. 14) For every subset A of S_1 such that $A \in$ the topology of K_1 holds A is simplex-like.

We now state the proposition

(33) S_1 is maximal iff $2^{\Omega_{(S_1)}} \cap$ the topology of $K_1 \subseteq$ the topology of S_1.

Let us consider X, K_1. Note that there exists a subsimplicial complex of K_1 which is maximal and strict.

We now state three propositions:

(34) Let S_2 be a subsimplicial complex of S_1. Suppose S_1 is maximal and S_2 is maximal. Then S_2 is a maximal subsimplicial complex of K_1.

(35) Let S_2 be a subsimplicial complex of S_1. If S_2 is a maximal subsimplicial complex of K_1, then S_2 is maximal.

(36) Let K_3, K_4 be maximal subsimplicial complexes of K_1. Suppose $\Omega_{(K_3)} = \Omega_{(K_4)}$. Then the topological structure of $K_3 =$ the topological structure of K_4.

Let us consider X, let K_1 be a subset-closed simplicial complex structure of X, and let A be a subset of K_1. Let us assume that $2^A \cap$ the topology of K_1 is finite-membered. The functor $K_1 \restriction A$ yields a maximal strict subsimplicial complex of K_1 and is defined as follows:

(Def. 15) $\Omega_{K_1 \restriction A} = A$.

In the sequel S_3 denotes a simplicial complex of X.

Let us consider X, S_3 and let A be a subset of S_3. Then $S_3 \restriction A$ is a maximal strict subsimplicial complex of S_3 and it can be characterized by the condition:

(Def. 16) $\Omega_{S_3 \restriction A} = A$.

The following four propositions are true:

(37) For every subset A of S_3 holds the topology of $S_3 \restriction A = 2^A \cap$ the topology of S_3.

(38) For all subsets A, B of S_3 and for every subset B' of $S_3 \restriction A$ such that $B' = B$ holds $S_3 \restriction A \restriction B' = S_3 \restriction B$.

(39) $S_3 \restriction \Omega_{(S_3)} =$ the topological structure of S_3.

(40) For all subsets A, B of S_3 such that $A \subseteq B$ holds $S_3 \restriction A$ is a subsimplicial complex of $S_3 \restriction B$.

Let us observe that every integer is finite.

6. The Skeleton of a Simplicial Complex

Let us consider X, K_1 and let i be a real number. The skeleton of K_1 and i yielding a simplicial complex structure of X is defined by the condition (Def. 17).

(Def. 17) The skeleton of K_1 and i = the complex of the subsets of the topology of K_1 with cardinality limited by $i+1$.

Let us consider X, K_1. Observe that the skeleton of K_1 and -1 is empty-membered. Let us consider i. Note that the skeleton of K_1 and i is finite-degree.

Let us consider X, let K_1 be an empty-membered simplicial complex structure of X, and let us consider i. One can check that the skeleton of K_1 and i is empty-membered.

Let us consider D, let K_2 be a non void subset-closed simplicial complex structure of D, and let us consider i. One can check that the skeleton of K_2 and i is non void.

One can prove the following proposition

(41) If $-1 \leq i_1 \leq i_2$, then the skeleton of K_1 and i_1 is a subsimplicial complex of the skeleton of K_1 and i_2.

Let us consider X, let K_1 be a subset-closed simplicial complex structure of X, and let us consider i. Then the skeleton of K_1 and i is a subsimplicial complex of K_1.

We now state several propositions:

(42) If K_1 is subset-closed and the skeleton of K_1 and i is empty-membered, then K_1 is empty-membered or $i = -1$.

(43) degree(the skeleton of K_1 and i) $\leq$ degree(K_1).

(44) If $-1 \leq i$, then degree(the skeleton of K_1 and i) $\leq i$.

(45) If $-1 \leq i$ and the skeleton of K_1 and i = the topological structure of K_1, then degree(K_1) $\leq i$.

(46) If K_1 is subset-closed and degree(K_1) $\leq i$, then the skeleton of K_1 and i = the topological structure of K_1.

In the sequel K is a non void subset-closed simplicial complex structure.

Let us consider K and let i be a real number. Let us assume that i is integer. A finite simplex of K is said to be a simplex of i and K if:

(Def. 18)(i) $\overline{\overline{\text{it}}} = i+1$ if $-1 \leq i \leq$ degree(K),

(ii) it is empty, otherwise.

Let us consider K. Note that every simplex of -1 and K is empty.

The following three propositions are true:

(47) For every simplex S of i and K such that S is non empty holds i is natural.

(48) Every finite simplex S of K is a simplex of $\overline{\overline{S}} - 1$ and K.

(49) Let K be a non void subset-closed simplicial complex structure of D, S be a non void subsimplicial complex of K, i be an integer, and A be a simplex of i and S. If A is non empty or $i \leq$ degree(S) or degree(S) = degree(K), then A is a simplex of i and K.

Let us consider K and let i be a real number. Let us assume that i is integer and $i \leq \mathrm{degree}(K)$. Let S be a simplex of i and K. A simplex of $\max(i-1,-1)$ and K is said to be a face of S if:

(Def. 19) It $\subseteq S$.

One can prove the following proposition

(50) Let S be a simplex of n and K. Suppose $n \leq \mathrm{degree}(K)$. Then X is a face of S if and only if there exists x such that $x \in S$ and $S \setminus \{x\} = X$.

7. The Subdivision of a Simplicial Complex

In the sequel P is a function.

Let us consider X, K_1, P. The functor subdivision(P, K_1) yields a strict simplicial complex structure of X and is defined by the conditions (Def. 20).

(Def. 20)(i) $\Omega_{\mathrm{subdivision}(P,K_1)} = \Omega_{(K_1)}$, and

(ii) for every subset A of subdivision(P, K_1) holds A is simplex-like iff there exists a $\subseteq$-linear finite simplex-like family S of subsets of K_1 such that $A = P^{\circ} S$.

Let us consider X, K_1, P. One can verify that subdivision(P, K_1) is subset-closed and finite-membered and has empty element.

Let us consider X, let K_1 be a void simplicial complex structure of X, and let us consider P. Observe that subdivision(P, K_1) is empty-membered.

The following propositions are true:

(51) $\mathrm{degree}(\mathrm{subdivision}(P, K_1)) \leq \mathrm{degree}(K_1) + 1$.

(52) If $\mathrm{dom}\, P$ has non empty elements, then $\mathrm{degree}(\mathrm{subdivision}(P, K_1)) \leq \mathrm{degree}(K_1)$.

Let us consider X, let K_1 be a finite-degree simplicial complex structure of X, and let us consider P. Note that subdivision(P, K_1) is finite-degree.

Let us consider X, let K_1 be a finite-vertices simplicial complex structure of X, and let us consider P. One can check that subdivision(P, K_1) is finite-vertices.

One can prove the following propositions:

(53) Let K_1 be a subset-closed simplicial complex structure of X and given P. Suppose that

(i) $\mathrm{dom}\, P$ has non empty elements, and

(ii) for every n such that $n \leq \mathrm{degree}(K_1)$ there exists a subset S of K_1 such that S is simplex-like and $\mathrm{Card}\, S = n + 1$ and $2^S_+ \subseteq \mathrm{dom}\, P$ and $P^{\circ} 2^S_+$ is a subset of K_1 and $P \restriction 2^S_+$ is one-to-one.

Then $\mathrm{degree}(\mathrm{subdivision}(P, K_1)) = \mathrm{degree}(K_1)$.

(54) If $Y \subseteq Z$, then subdivision$(P \restriction Y, K_1)$ is a subsimplicial complex of subdivision$(P \restriction Z, K_1)$.

(55) If $\mathrm{dom}\, P \cap$ the topology of $K_1 \subseteq Y$, then subdivision$(P{\restriction}Y, K_1) =$ subdivision(P, K_1).

(56) If $Y \subseteq Z$, then subdivision$(Y{\restriction}P, K_1)$ is a subsimplicial complex of subdivision$(Z{\restriction}P, K_1)$.

(57) If P°(the topology of K_1) $\subseteq Y$, then subdivision$(Y{\restriction}P, K_1) =$ subdivision(P, K_1).

(58) subdivision(P, S_1) is a subsimplicial complex of subdivision(P, K_1).

(59) For every subset A of subdivision(P, K_1) such that $\mathrm{dom}\, P \subseteq$ the topology of S_1 and $A = \Omega_{(S_1)}$ holds subdivision$(P, S_1) =$ subdivision$(P, K_1){\restriction}A$.

(60) Let K_3, K_4 be simplicial complex structures of X. Suppose the topological structure of K_3 = the topological structure of K_4. Then subdivision$(P, K_3) =$ subdivision(P, K_4).

Let us consider X, K_1, P, n. The functor subdivision(n, P, K_1) yielding a simplicial complex structure of X is defined by the condition (Def. 21).

(Def. 21) There exists a function F such that

(i) $F(0) = K_1$,

(ii) $F(n) =$ subdivision(n, P, K_1),

(iii) $\mathrm{dom}\, F = \mathbb{N}$, and

(iv) for every k and for every simplicial complex structure K_1' of X such that $K_1' = F(k)$ holds $F(k+1) =$ subdivision(P, K_1').

Next we state several propositions:

(61) subdivision$(0, P, K_1) = K_1$.

(62) subdivision$(1, P, K_1) =$ subdivision(P, K_1).

(63) For every natural number n_1 such that $n_1 = n + k$ holds subdivision$(n_1, P, K_1) =$ subdivision$(n, P,$ subdivision$(k, P, K_1))$.

(64) $\Omega_{\mathrm{subdivision}(n,P,K_1)} = \Omega_{(K_1)}$.

(65) subdivision(n, P, S_1) is a subsimplicial complex of subdivision(n, P, K_1).

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