

# Free Magmas

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**Summary.** This article introduces the free magma  $M(X)$  constructed on a set  $X$  [6]. Then, we formalize some theorems about  $M(X)$ : if  $f$  is a function from the set  $X$  to a magma  $N$ , the free magma  $M(X)$  has a unique extension of  $f$  to a morphism of  $M(X)$  into  $N$  and every magma is isomorphic to a magma generated by a set  $X$  under a set of relators on  $M(X)$ . In doing it, the article defines the stable subset under the law of composition of a magma, the submagma, the equivalence relation compatible with the law of composition and the equivalence kernel of a function. We also introduce some schemes on the recursive function.

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The terminology and notation used here have been introduced in the following articles: [19], [12], [7], [2], [14], [4], [8], [9], [17], [15], [1], [3], [10], [5], [20], [21], [13], [18], [16], and [11].

## 1. PRELIMINARIES

Let  $X$  be a set, let  $f$  be a function from  $\mathbb{N}$  into  $X$ , and let  $n$  be a natural number. Observe that  $f \upharpoonright n$  is transfinite sequence-like.

Let  $X, x$  be sets. The 0-sequence  $x_X(x)$  yielding a finite 0-sequence of  $X$  is defined as follows:

(Def. 1) The 0-sequence  $x_X(x) = \begin{cases} x, & \text{if } x \text{ is a finite 0-sequence of } X, \\ \langle \rangle_X, & \text{otherwise.} \end{cases}$

Let  $X$  be a non empty set, let  $f$  be a function from  $X^\omega$  into  $X$ , and let  $c$  be a finite 0-sequence of  $X$ . Then  $f(c)$  is an element of  $X$ .

One can prove the following proposition

- (1) For all sets  $X, Y, Z$  such that  $Y \subseteq$  the universe of  $X$  and  $Z \subseteq$  the universe of  $X$  holds  $Y \times Z \subseteq$  the universe of  $X$ .

In this article we present several logical schemes. The scheme *FuncRecursiveUniq* deals with a unary functor  $\mathcal{F}$  yielding a set and functions  $\mathcal{A}$ ,  $\mathcal{B}$ , and states that:

$$\mathcal{A} = \mathcal{B}$$

provided the parameters satisfy the following conditions:

- $\text{dom } \mathcal{A} = \mathbb{N}$  and for every natural number  $n$  holds  $\mathcal{A}(n) = \mathcal{F}(\mathcal{A}\upharpoonright n)$ ,  
and
- $\text{dom } \mathcal{B} = \mathbb{N}$  and for every natural number  $n$  holds  $\mathcal{B}(n) = \mathcal{F}(\mathcal{B}\upharpoonright n)$ .

The scheme *FuncRecursiveExist* deals with a unary functor  $\mathcal{F}$  yielding a set, and states that:

There exists a function  $f$  such that  $\text{dom } f = \mathbb{N}$  and for every natural number  $n$  holds  $f(n) = \mathcal{F}(f\upharpoonright n)$

for all values of the parameter.

The scheme *FuncRecursiveUniqu2* deals with a non empty set  $\mathcal{A}$ , a unary functor  $\mathcal{F}$  yielding an element of  $\mathcal{A}$ , and functions  $\mathcal{B}$ ,  $\mathcal{C}$  from  $\mathbb{N}$  into  $\mathcal{A}$ , and states that:

$$\mathcal{B} = \mathcal{C}$$

provided the parameters meet the following requirements:

- For every element  $n$  of  $\mathbb{N}$  holds  $\mathcal{B}(n) = \mathcal{F}(\mathcal{B}\upharpoonright n)$ , and
- For every element  $n$  of  $\mathbb{N}$  holds  $\mathcal{C}(n) = \mathcal{F}(\mathcal{C}\upharpoonright n)$ .

The scheme *FuncRecursiveExist2* deals with a non empty set  $\mathcal{A}$  and a unary functor  $\mathcal{F}$  yielding an element of  $\mathcal{A}$ , and states that:

There exists a function  $f$  from  $\mathbb{N}$  into  $\mathcal{A}$  such that for every natural number  $n$  holds  $f(n) = \mathcal{F}(f\upharpoonright n)$

for all values of the parameters.

Let  $f, g$  be functions. We say that  $f$  extends  $g$  if and only if:

(Def. 2)  $\text{dom } g \subseteq \text{dom } f$  and  $f \approx g$ .

Let us note that there exists a multiplicative magma which is empty.

## 2. EQUIVALENCE RELATIONS AND RELATORS

Let  $M$  be a multiplicative magma and let  $R$  be an equivalence relation of  $M$ . We say that  $R$  is compatible if and only if:

(Def. 3) For all elements  $v, v', w, w'$  of  $M$  such that  $v \in [v']_R$  and  $w \in [w']_R$  holds  $v \cdot w \in [v' \cdot w']_R$ .

Let  $M$  be a multiplicative magma. Observe that  $\nabla_{\text{the carrier of } M}$  is compatible.

Let  $M$  be a multiplicative magma. Observe that there exists an equivalence relation of  $M$  which is compatible.

One can prove the following proposition

- (2) Let  $M$  be a multiplicative magma and  $R$  be an equivalence relation of  $M$ . Then  $R$  is compatible if and only if for all elements  $v, v', w, w'$  of  $M$  such that  $[v]_R = [v']_R$  and  $[w]_R = [w']_R$  holds  $[v \cdot w]_R = [v' \cdot w']_R$ .

Let  $M$  be a multiplicative magma and let  $R$  be a compatible equivalence relation of  $M$ . The functor  $\circ_R$  yielding a binary operation on Classes  $R$  is defined as follows:

- (Def. 4)(i) For all elements  $x, y$  of Classes  $R$  and for all elements  $v, w$  of  $M$  such that  $x = [v]_R$  and  $y = [w]_R$  holds  $(\circ_R)(x, y) = [v \cdot w]_R$  if  $M$  is non empty,  
(ii)  $\circ_R = \emptyset$ , otherwise.

Let  $M$  be a multiplicative magma and let  $R$  be a compatible equivalence relation of  $M$ . The functor  $^M/R$  yielding a multiplicative magma is defined as follows:

- (Def. 5)  $^M/R = \langle \text{Classes } R, \circ_R \rangle$ .

Let  $M$  be a multiplicative magma and let  $R$  be a compatible equivalence relation of  $M$ . Observe that  $^M/R$  is strict.

Let  $M$  be a non empty multiplicative magma and let  $R$  be a compatible equivalence relation of  $M$ . One can check that  $^M/R$  is non empty.

Let  $M$  be a non empty multiplicative magma and let  $R$  be a compatible equivalence relation of  $M$ . The canonical homomorphism onto cosets of  $R$  yields a function from  $M$  into  $^M/R$  and is defined by:

- (Def. 6) For every element  $v$  of  $M$  holds (the canonical homomorphism onto cosets of  $R$ )( $v$ ) =  $[v]_R$ .

Let  $M$  be a non empty multiplicative magma and let  $R$  be a compatible equivalence relation of  $M$ . Note that the canonical homomorphism onto cosets of  $R$  is multiplicative.

Let  $M$  be a non empty multiplicative magma and let  $R$  be a compatible equivalence relation of  $M$ . Note that the canonical homomorphism onto cosets of  $R$  is onto.

Let  $M$  be a multiplicative magma. A function is called a relators of  $M$  if:

- (Def. 7)  $\text{rng } r \subseteq (\text{the carrier of } M) \times (\text{the carrier of } M)$ .

Let  $M$  be a multiplicative magma and let  $r$  be a relators of  $M$ . The equivalence relation of  $r$  yielding an equivalence relation of  $M$  is defined by the condition (Def. 8).

- (Def. 8) The equivalence relation of  $r = \bigcap \{R; R \text{ ranges over compatible equivalence relations of } M: \bigwedge_{i:\text{set}} \bigwedge_{v,w:\text{element of } M} (i \in \text{dom } r \wedge r(i) = \langle v, w \rangle \Rightarrow v \in [w]_R)\}$ .

Next we state the proposition

- (3) Let  $M$  be a multiplicative magma,  $r$  be a relators of  $M$ , and  $R$  be a compatible equivalence relation of  $M$ . Suppose that for every set  $i$  and

for all elements  $v, w$  of  $M$  such that  $i \in \text{dom } r$  and  $r(i) = \langle v, w \rangle$  holds  $v \in [w]_R$ . Then the equivalence relation of  $r \subseteq R$ .

Let  $M$  be a multiplicative magma and let  $r$  be a relators of  $M$ . Note that the equivalence relation of  $r$  is compatible.

Let  $X, Y$  be sets and let  $f$  be a function from  $X$  into  $Y$ . The equivalence kernel of  $f$  yielding an equivalence relation of  $X$  is defined as follows:

(Def. 9) For all sets  $x, y$  holds  $\langle x, y \rangle \in$  the equivalence kernel of  $f$  iff  $x, y \in X$  and  $f(x) = f(y)$ .

In the sequel  $M, N$  are non empty multiplicative magmas and  $f$  is a function from  $M$  into  $N$ .

The following propositions are true:

- (4) If  $f$  is multiplicative, then the equivalence kernel of  $f$  is compatible.
- (5) Suppose  $f$  is multiplicative. Then there exists a relators  $r$  of  $M$  such that the equivalence kernel of  $f =$  the equivalence relation of  $r$ .

### 3. SUBMAGMAS AND STABLE SUBSETS

Let  $M$  be a multiplicative magma. A multiplicative magma is said to be a submagma of  $M$  if it satisfies the conditions (Def. 10).

(Def. 10)(i) The carrier of it  $\subseteq$  the carrier of  $M$ , and  
(ii) the multiplication of it = (the multiplication of  $M$ )  $\upharpoonright$  (the carrier of it).

Let  $M$  be a multiplicative magma. One can check that there exists a submagma of  $M$  which is strict.

Let  $M$  be a non empty multiplicative magma. Note that there exists a submagma of  $M$  which is non empty.

In the sequel  $M$  denotes a multiplicative magma and  $N, K$  denote submagmas of  $M$ .

One can prove the following propositions:

- (6) Suppose  $N$  is a submagma of  $K$  and  $K$  is a submagma of  $N$ . Then the multiplicative magma of  $N =$  the multiplicative magma of  $K$ .
- (7) Suppose the carrier of  $N =$  the carrier of  $M$ . Then the multiplicative magma of  $N =$  the multiplicative magma of  $M$ .

Let  $M$  be a multiplicative magma and let  $A$  be a subset of  $M$ . We say that  $A$  is stable if and only if:

(Def. 11) For all elements  $v, w$  of  $M$  such that  $v, w \in A$  holds  $v \cdot w \in A$ .

Let  $M$  be a multiplicative magma. One can check that there exists a subset of  $M$  which is stable.

We now state the proposition

- (8) The carrier of  $N$  is a stable subset of  $M$ .

Let  $M$  be a multiplicative magma and let  $N$  be a submagma of  $M$ . Note that the carrier of  $N$  is stable.

We now state the proposition

- (9) Let  $F$  be a function. Suppose that for every set  $i$  such that  $i \in \text{dom } F$  holds  $F(i)$  is a stable subset of  $M$ . Then  $\bigcap F$  is a stable subset of  $M$ .

For simplicity, we adopt the following convention:  $M, N$  are non empty multiplicative magmas,  $A$  is a subset of  $M$ ,  $f, g$  are functions from  $M$  into  $N$ ,  $X$  is a stable subset of  $M$ , and  $Y$  is a stable subset of  $N$ .

Next we state four propositions:

- (10)  $A$  is stable iff  $A \cdot A \subseteq A$ .  
 (11) If  $f$  is multiplicative, then  $f^\circ X$  is a stable subset of  $N$ .  
 (12) If  $f$  is multiplicative, then  $f^{-1}(Y)$  is a stable subset of  $M$ .  
 (13) If  $f$  is multiplicative and  $g$  is multiplicative, then  $\{v \in M: f(v) = g(v)\}$  is a stable subset of  $M$ .

Let  $M$  be a multiplicative magma and let  $A$  be a stable subset of  $M$ . The multiplication induced by  $A$  yields a binary operation on  $A$  and is defined by:

(Def. 12) The multiplication induced by  $A = (\text{the multiplication of } M) \upharpoonright A$ .

Let  $M$  be a multiplicative magma and let  $A$  be a subset of  $M$ . The submagma generated by  $A$  yielding a strict submagma of  $M$  is defined by the conditions (Def. 13).

- (Def. 13)(i)  $A \subseteq$  the carrier of the submagma generated by  $A$ , and  
 (ii) for every strict submagma  $N$  of  $M$  such that  $A \subseteq$  the carrier of  $N$  holds the submagma generated by  $A$  is a submagma of  $N$ .

We now state the proposition

- (14) Let  $M$  be a multiplicative magma and  $A$  be a subset of  $M$ . Then  $A$  is empty if and only if the submagma generated by  $A$  is empty.

Let  $M$  be a multiplicative magma and let  $A$  be an empty subset of  $M$ . Note that the submagma generated by  $A$  is empty.

The following proposition is true

- (15) Let  $M, N$  be non empty multiplicative magmas,  $f$  be a function from  $M$  into  $N$ , and  $X$  be a subset of  $M$ . Suppose  $f$  is multiplicative. Then  $f^\circ(\text{the carrier of the submagma generated by } X) = \text{the carrier of the submagma generated by } f^\circ X$ .

## 4. FREE MAGMAS

Let  $X$  be a set. The free magma sequence of  $X$  yielding a function from  $\mathbb{N}$  into  $2^{\text{the universe of } X \cup \mathbb{N}}$  is defined by the conditions (Def. 14).

- (Def. 14)(i) (The free magma sequence of  $X$ )(0) =  $\emptyset$ ,  
(ii) (the free magma sequence of  $X$ )(1) =  $X$ , and  
(iii) for every natural number  $n$  such that  $n \geq 2$  there exists a finite sequence  $f_1$  such that  $\text{len } f_1 = n - 1$  and for every natural number  $p$  such that  $p \geq 1$  and  $p \leq n - 1$  holds  $f_1(p) = (\text{the free magma sequence of } X)(p) \times (\text{the free magma sequence of } X)(n - p)$  and  $(\text{the free magma sequence of } X)(n) = \bigcup \text{disjoint } f_1$ .

Let  $X$  be a set and let  $n$  be a natural number. The functor  $M_n(X)$  is defined by:

- (Def. 15)  $M_n(X) = (\text{the free magma sequence of } X)(n)$ .

Let  $X$  be a non empty set and let  $n$  be a non zero natural number. Observe that  $M_n(X)$  is non empty.

In the sequel  $X$  is a set.

We now state four propositions:

- (16)  $M_0(X) = \emptyset$ .  
(17)  $M_1(X) = X$ .  
(18)  $M_2(X) = X \times X \times \{1\}$ .  
(19)  $M_3(X) = X \times (X \times X \times \{1\}) \times \{1\} \cup X \times X \times \{1\} \times X \times \{2\}$ .

We adopt the following convention:  $x, y, Y$  are sets and  $n, m, p$  are elements of  $\mathbb{N}$ .

One can prove the following propositions:

- (20) Suppose  $n \geq 2$ . Then there exists a finite sequence  $f_1$  such that  $\text{len } f_1 = n - 1$  and for every  $p$  such that  $p \geq 1$  and  $p \leq n - 1$  holds  $f_1(p) = M_p(X) \times M_{n-p}(X)$  and  $M_n(X) = \bigcup \text{disjoint } f_1$ .  
(21) Suppose  $n \geq 2$  and  $x \in M_n(X)$ . Then there exist  $p, m$  such that  $x_2 = p$  and  $1 \leq p \leq n - 1$  and  $(x_1)_1 \in M_p(X)$  and  $(x_1)_2 \in M_m(X)$  and  $n = p + m$  and  $x \in M_p(X) \times M_m(X) \times \{p\}$ .  
(22) If  $x \in M_n(X)$  and  $y \in M_m(X)$ , then  $\langle \langle x, y \rangle, n \rangle \in M_{n+m}(X)$ .  
(23) If  $X \subseteq Y$ , then  $M_n(X) \subseteq M_n(Y)$ .

Let  $X$  be a set. The carrier of free magma on  $X$  is defined as follows:

- (Def. 16) The carrier of free magma on  $X = \bigcup \text{disjoint}((\text{the free magma sequence of } X) \upharpoonright \mathbb{N}^+)$ .

One can prove the following proposition

- (24)  $X = \emptyset$  iff the carrier of free magma on  $X = \emptyset$ .

Let  $X$  be an empty set. Observe that the carrier of free magma on  $X$  is empty.

Let  $X$  be a non empty set. One can verify that the carrier of free magma on  $X$  is non empty. Let  $w$  be an element of the carrier of free magma on  $X$ . Observe that  $w_2$  is non zero and natural.

We now state four propositions:

- (25) For every non empty set  $X$  and for every element  $w$  of the carrier of free magma on  $X$  holds  $w \in M_{w_2}(X) \times \{w_2\}$ .
- (26) Let  $X$  be a non empty set and  $v, w$  be elements of the carrier of free magma on  $X$ . Then  $\langle \langle \langle v_1, w_1 \rangle, v_2 \rangle, v_2 + w_2 \rangle$  is an element of the carrier of free magma on  $X$ .
- (27) If  $X \subseteq Y$ , then the carrier of free magma on  $X \subseteq$  the carrier of free magma on  $Y$ .
- (28) If  $n > 0$ , then  $M_n(X) \times \{n\} \subseteq$  the carrier of free magma on  $X$ .

Let  $X$  be a set. The multiplication of free magma on  $X$  yields a binary operation on the carrier of free magma on  $X$  and is defined as follows:

- (Def. 17)(i) For all elements  $v, w$  of the carrier of free magma on  $X$  and for all  $n, m$  such that  $n = v_2$  and  $m = w_2$  holds (the multiplication of free magma on  $X$ )( $v, w$ ) =  $\langle \langle \langle v_1, w_1 \rangle, v_2 \rangle, n + m \rangle$  if  $X$  is non empty,
- (ii) the multiplication of free magma on  $X = \emptyset$ , otherwise.

Let  $X$  be a set. The functor  $M(X)$  yields a multiplicative magma and is defined by:

- (Def. 18)  $M(X) = \langle$ the carrier of free magma on  $X$ , the multiplication of free magma on  $X \rangle$ .

Let  $X$  be a set. Note that  $M(X)$  is strict.

Let  $X$  be an empty set. One can verify that  $M(X)$  is empty.

Let  $X$  be a non empty set. Note that  $M(X)$  is non empty. Let  $w$  be an element of  $M(X)$ . One can verify that  $w_2$  is non zero and natural.

The following proposition is true

- (29) For every set  $X$  and for every subset  $S$  of  $X$  holds  $M(S)$  is a submagma of  $M(X)$ .

Let  $X$  be a set and let  $w$  be an element of  $M(X)$ . The functor  $\text{length } w$  yields a natural number and is defined by:

- (Def. 19)  $\text{length } w = \begin{cases} w_2, & \text{if } X \text{ is non empty,} \\ 0, & \text{otherwise.} \end{cases}$

One can prove the following proposition

- (30)  $X = \{w_1; w \text{ ranges over elements of } M(X): \text{length } w = 1\}$ .

In the sequel  $v, v_1, v_2, w, w_1, w_2$  denote elements of  $M(X)$ .

One can prove the following propositions:

- (31) If  $X$  is non empty, then  $v \cdot w = \langle \langle v_1, w_1 \rangle, v_2 \rangle$ ,  $\text{length } v + \text{length } w$ .
- (32) If  $X$  is non empty, then  $v = \langle v_1, v_2 \rangle$  and  $\text{length } v \geq 1$ .
- (33)  $\text{length}(v \cdot w) = \text{length } v + \text{length } w$ .
- (34) If  $\text{length } w \geq 2$ , then there exist  $w_1, w_2$  such that  $w = w_1 \cdot w_2$  and  $\text{length } w_1 < \text{length } w$  and  $\text{length } w_2 < \text{length } w$ .
- (35) If  $v_1 \cdot v_2 = w_1 \cdot w_2$ , then  $v_1 = w_1$  and  $v_2 = w_2$ .

Let  $X$  be a set and let  $n$  be a natural number. The  $n$ -canonical image of  $X$  yields a function from  $M_n(X)$  into  $M(X)$  and is defined as follows:

- (Def. 20)(i) For every  $x$  such that  $x \in \text{dom}$  (the  $n$ -canonical image of  $X$ ) holds  
 (the  $n$ -canonical image of  $X$ )( $x$ ) =  $\langle x, n \rangle$  if  $n > 0$ ,
- (ii) the  $n$ -canonical image of  $X = \emptyset$ , otherwise.

Let  $X$  be a set and let  $n$  be a natural number. Observe that the  $n$ -canonical image of  $X$  is one-to-one.

Let  $X$  be a non empty set. Observe that the 1-canonical image of  $X$

In the sequel  $X, Y, Z$  are non empty sets.

Next we state three propositions:

- (36) For every subset  $A$  of  $M(X)$  such that  $A = (\text{the 1-canonical image of } X)^\circ X$  holds  $M(X) = \text{the submagma generated by } A$ .
- (37) Let  $R$  be a compatible equivalence relation of  $M(X)$ . Then  $M(X)/R = \text{the submagma generated by } (\text{the canonical homomorphism onto cosets of } R)^\circ (\text{the 1-canonical image of } X)^\circ X$ .
- (38) For every function  $f$  from  $X$  into  $Y$  holds  $(\text{the 1-canonical image of } Y) \cdot f$  is a function from  $X$  into  $M(Y)$ .

Let  $X$  be a non empty set, let  $M$  be a non empty multiplicative magma, let  $n, m$  be non zero natural numbers, let  $f$  be a function from  $M_n(X)$  into  $M$ , and let  $g$  be a function from  $M_m(X)$  into  $M$ . The functor  $f \times g$  yielding a function from  $M_n(X) \times M_m(X) \times \{n\}$  into  $M$  is defined by the condition (Def. 21).

- (Def. 21) Let  $x$  be an element of  $M_n(X) \times M_m(X) \times \{n\}$ ,  $y$  be an element of  $M_n(X)$ , and  $z$  be an element of  $M_m(X)$ . If  $y = (x_1)_1$  and  $z = (x_1)_2$ , then  
 $(f \times g)(x) = f(y) \cdot g(z)$ .

In the sequel  $M$  is a non empty multiplicative magma.

One can prove the following propositions:

- (39) Let  $f$  be a function from  $X$  into  $M$ . Then there exists a function  $h$  from  $M(X)$  into  $M$  such that  $h$  is multiplicative and  $h$  extends  $f \cdot (\text{the 1-canonical image of } X)^{-1}$ .
- (40) Let  $f$  be a function from  $X$  into  $M$  and  $h, g$  be functions from  $M(X)$  into  $M$ . Suppose that
- (i)  $h$  is multiplicative,
  - (ii)  $h$  extends  $f \cdot (\text{the 1-canonical image of } X)^{-1}$ ,
  - (iii)  $g$  is multiplicative, and

(iv)  $g$  extends  $f \cdot (\text{the 1-canonical image of } X)^{-1}$ .

Then  $h = g$ .

For simplicity, we adopt the following rules:  $M, N$  are non empty multiplicative magmas,  $f$  is a function from  $M$  into  $N$ ,  $H$  is a non empty submagma of  $N$ , and  $R$  is a compatible equivalence relation of  $M$ .

We now state three propositions:

- (41) Suppose  $f$  is multiplicative and the carrier of  $H = \text{rng } f$  and  $R =$  the equivalence kernel of  $f$ . Then there exists a function  $g$  from  $M/R$  into  $H$  such that  $f = g \cdot$  the canonical homomorphism onto cosets of  $R$  and  $g$  is bijective and multiplicative.
- (42) Let  $g_1, g_2$  be functions from  $M/R$  into  $N$ . Suppose  $g_1 \cdot$  the canonical homomorphism onto cosets of  $R = g_2 \cdot$  the canonical homomorphism onto cosets of  $R$ . Then  $g_1 = g_2$ .
- (43) Let  $M$  be a non empty multiplicative magma. Then there exists a non empty set  $X$  and there exists a relators  $r$  of  $M(X)$  such that there exists a function from  $M(X)/\text{the equivalence relation of } r$  into  $M$  which is bijective and multiplicative.

Let  $X, Y$  be non empty sets and let  $f$  be a function from  $X$  into  $Y$ . The functor  $\mathbf{M}(f)$  yields a function from  $M(X)$  into  $M(Y)$  and is defined by:

(Def. 22)  $\mathbf{M}(f)$  is multiplicative and  $\mathbf{M}(f)$  extends  $(\text{the 1-canonical image of } Y) \cdot f \cdot (\text{the 1-canonical image of } X)^{-1}$ .

Let  $X, Y$  be non empty sets and let  $f$  be a function from  $X$  into  $Y$ . One can verify that  $\mathbf{M}(f)$  is multiplicative.

In the sequel  $f$  denotes a function from  $X$  into  $Y$  and  $g$  denotes a function from  $Y$  into  $Z$ .

Next we state several propositions:

- (44)  $\mathbf{M}(g \cdot f) = \mathbf{M}(g) \cdot \mathbf{M}(f)$ .
- (45) For every function  $g$  from  $X$  into  $Z$  such that  $Y \subseteq Z$  and  $f = g$  holds  $\mathbf{M}(f) = \mathbf{M}(g)$ .
- (46)  $\mathbf{M}(\text{id}_X) = \text{id}_{\text{dom } \mathbf{M}(f)}$ .
- (47) If  $f$  is one-to-one, then  $\mathbf{M}(f)$  is one-to-one.
- (48) If  $f$  is onto, then  $\mathbf{M}(f)$  is onto.

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