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Banach Algebra of Continuous Functionals and the Space of Real-Valued Continuous Functionals with Bounded Support[1]

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Summary. In this article, we give a definition of a functional space which is constructed from all continuous functions defined on a compact topological space. We prove that this functional space is a Banach algebra. Next, we give a definition of a function space which is constructed from all real-valued continuous functions with bounded support. We prove that this function space is a real normed space.

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The notation and terminology used here have been introduced in the following papers: [2], [15], [7], [17], [16], [10], [3], [18], [14], [13], [12], [1], [4], [11], [6], [8], [19], [20], [9], and [5].

1. Banach Algebra of Continuous Functionals

Let X be a 1-sorted structure and let y be a real number. The functor $X \longmapsto y$ yielding a real map of X is defined as follows:

(Def. 1) $X \longmapsto y = (\text{the carrier of } X) \longmapsto y$.

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Let X be a topological space and let y be a real number. Note that $X \longmapsto y$ is continuous.

Next we state the proposition

(1) Let X be a non empty topological space and f be a real map of X. Then f is continuous if and only if for every point x of X and for every subset V of $\mathbb{R}$ such that $f(x) \in V$ and V is open there exists a subset W of X such that $x \in W$ and W is open and $f^{\circ}W \subseteq V$.

In the sequel X denotes a non empty topological space.

Let us consider X. The functor $\mathrm{C}(X;\mathbb{R})$ yielding a subset of RAlgebra (the carrier of X) is defined by:

(Def. 2) $\mathrm{C}(X;\mathbb{R}) = \{f : f$ ranges over continuous real maps of $X\}$.

Let us consider X. Observe that $\mathrm{C}(X;\mathbb{R})$ is non empty.

Let us consider X. One can verify that $\mathrm{C}(X;\mathbb{R})$ is additively-linearly-closed and multiplicatively-closed.

Let X be a non empty topological space. The functor $\mathrm{C_A}(X;\mathbb{R})$ yielding an algebra structure is defined by the condition (Def. 3).

(Def. 3) $\mathrm{C_A}(X;\mathbb{R}) = \langle \mathrm{C}(X;\mathbb{R}), \mathrm{mult}(\mathrm{C}(X;\mathbb{R}), \mathrm{RAlgebra}\,(\text{the carrier of } X)),$ $\mathrm{Add}(\mathrm{C}(X;\mathbb{R}), \mathrm{RAlgebra}\,(\text{the carrier of } X)), \mathrm{Mult}(\mathrm{C}(X;\mathbb{R}), \mathrm{RAlgebra}\,(\text{the carrier of } X)), \mathrm{One}(\mathrm{C}(X;\mathbb{R}), \mathrm{RAlgebra}\,(\text{the carrier of } X)), \mathrm{Zero}(\mathrm{C}(X;\mathbb{R}),$ $\mathrm{RAlgebra}\,(\text{the carrier of } X))\rangle$.

One can prove the following proposition

(2) $\mathrm{C_A}(X;\mathbb{R})$ is a subalgebra of RAlgebra (the carrier of X).

Let us consider X. Note that $\mathrm{C_A}(X;\mathbb{R})$ is strict and non empty.

Let us consider X. Observe that $\mathrm{C_A}(X;\mathbb{R})$ is Abelian, add-associative, right zeroed, right complementable, vector distributive, scalar distributive, scalar associative, scalar unital, commutative, associative, right unital, right distributive, vector distributive, scalar distributive, scalar associative, and vector associative.

We use the following convention: F, G, H denote vectors of $\mathrm{C_A}(X;\mathbb{R})$, g, h denote real maps of X, and a denotes a real number.

One can prove the following propositions:

(3) Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F + G$ if and only if for every element x of the carrier of X holds $h(x) = f(x) + g(x)$.

(4) If $f = F$ and $g = G$, then $G = a \cdot F$ iff for every element x of X holds $g(x) = a \cdot f(x)$.

(5) Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F \cdot G$ if and only if for every element x of the carrier of X holds $h(x) = f(x) \cdot g(x)$.

(6) $0_{\mathrm{C_A}(X;\mathbb{R})} = X \longmapsto 0$.

(7) $\mathbf{1}_{\mathrm{C_A}(X;\mathbb{R})} = X \longmapsto 1$.

In the sequel X denotes a compact non empty topological space and f, g, h denote real maps of X.

We now state two propositions:

(8) Let A be an algebra and A_1, A_2 be subalgebras of A. Suppose the carrier of $A_1 \subseteq$ the carrier of A_2. Then A_1 is a subalgebra of A_2.

(9) $\mathrm{C_A}(X;\mathbb{R})$ is a subalgebra of the $\mathbb{R}$-algebra of bounded functions on the carrier of X.

Let us consider X. The functor $||\cdot||_{C(X;\mathbb{R})}$ yielding a function from $\mathrm{C}(X;\mathbb{R})$ into $\mathbb{R}$ is defined as follows:

(Def. 4) $||\cdot||_{C(X;\mathbb{R})} = \mathrm{BoundedFunctionsNorm}\,(\text{the carrier of } X) \restriction \mathrm{C}(X;\mathbb{R})$.

Let us consider X. The functor $\mathrm{C_{NA}}(X;\mathbb{R})$ yielding a normed algebra structure is defined by the condition (Def. 5).

(Def. 5) $\mathrm{C_{NA}}(X;\mathbb{R}) = \langle \mathrm{C}(X;\mathbb{R}), \mathrm{mult}(\mathrm{C}(X;\mathbb{R}), \mathrm{RAlgebra}\,(\text{the carrier of } X)), \mathrm{Add}(\mathrm{C}(X;\mathbb{R}), \mathrm{RAlgebra}\,(\text{the carrier of } X)), \mathrm{Mult}(\mathrm{C}(X;\mathbb{R}), \mathrm{RAlgebra}\,(\text{the carrier of } X)), \mathrm{One}(\mathrm{C}(X;\mathbb{R}), \mathrm{RAlgebra}\,(\text{the carrier of } X)), \mathrm{Zero}(\mathrm{C}(X;\mathbb{R}), \mathrm{RAlgebra}\,(\text{the carrier of } X)), ||\cdot||_{C(X;\mathbb{R})}\rangle$.

Let us consider X. Observe that $\mathrm{C_{NA}}(X;\mathbb{R})$ is strict and non empty.

Let us consider X. Note that $\mathrm{C_{NA}}(X;\mathbb{R})$ is unital.

Next we state the proposition

(10) Let W be a normed algebra structure and V be an algebra. If the algebra structure of $W = V$, then W is an algebra.

In the sequel F, G, H denote points of $\mathrm{C_{NA}}(X;\mathbb{R})$.

Let us consider X. Note that $\mathrm{C_{NA}}(X;\mathbb{R})$ is Abelian, add-associative, right zeroed, right complementable, commutative, associative, right unital, right distributive, vector distributive, scalar distributive, scalar associative, and vector associative.

We now state the proposition

(11) $(\mathrm{Mult}(\mathrm{C}(X;\mathbb{R}), \mathrm{RAlgebra}\,(\text{the carrier of } X)))(1, F) = F$.

Let us consider X. Note that $\mathrm{C_{NA}}(X;\mathbb{R})$ is vector distributive, scalar distributive, scalar associative, and scalar unital.

We now state several propositions:

(12) $X \longmapsto 0 = 0_{\mathrm{C_{NA}}(X;\mathbb{R})}$.

(13) $0 \leq \|F\|$.

(14) $0 = \|(0_{\mathrm{C_{NA}}(X;\mathbb{R})})\|$.

(15) If $f = F$ and $g = G$ and $h = H$, then $H = F + G$ iff for every element x of X holds $h(x) = f(x) + g(x)$.

(16) If $f = F$ and $g = G$, then $G = a \cdot F$ iff for every element x of X holds $g(x) = a \cdot f(x)$.

(17) If $f = F$ and $g = G$ and $h = H$, then $H = F \cdot G$ iff for every element x of X holds $h(x) = f(x) \cdot g(x)$.

(18) $\|F\| = 0$ iff $F = 0_{\mathrm{C_{NA}}(X;\mathbb{R})}$ and $\|a \cdot F\| = |a| \cdot \|F\|$ and $\|F + G\| \leq \|F\| + \|G\|$.

Let us consider X. One can check that $\mathrm{C_{NA}}(X;\mathbb{R})$ is reflexive, discernible, and real normed space-like.

Next we state four propositions:

(19) If $f = F$ and $g = G$ and $h = H$, then $H = F - G$ iff for every element x of X holds $h(x) = f(x) - g(x)$.

(20) Let X be a real Banach space, Y be a subset of X, and s_1 be a sequence of X. Suppose Y is closed and $\mathrm{rng}\, s_1 \subseteq Y$ and s_1 is Cauchy sequence by norm. Then s_1 is convergent and $\lim s_1 \in Y$.

(21) Let Y be a subset of the $\mathbb{R}$-normed algebra of bounded functions on the carrier of X. If $Y = \mathrm{C}(X;\mathbb{R})$, then Y is closed.

(22) For every sequence s_1 of $\mathrm{C_{NA}}(X;\mathbb{R})$ such that s_1 is Cauchy sequence by norm holds s_1 is convergent.

Let us consider X. One can verify that $\mathrm{C_{NA}}(X;\mathbb{R})$ is complete.

Let us consider X. Observe that $\mathrm{C_{NA}}(X;\mathbb{R})$ is Banach Algebra-like.

2. Some Properties of Support

Next we state three propositions:

(23) For every non empty topological space X and for all real maps f, g of X holds $\mathrm{support}(f + g) \subseteq \mathrm{support}\, f \cup \mathrm{support}\, g$.

(24) For every non empty topological space X and for every real number a and for every real map f of X holds $\mathrm{support}(a\, f) \subseteq \mathrm{support}\, f$.

(25) For every non empty topological space X and for all real maps f, g of X holds $\mathrm{support}(f\, g) \subseteq \mathrm{support}\, f \cup \mathrm{support}\, g$.

3. The Space of Real-valued Continuous Functionals with Bounded Support

Let X be a non empty topological space. The functor $\mathrm{C}_0(X)$ yielding a non empty subset of $\mathbb{R}_{\mathbb{R}}^{\text{the carrier of } X}$ is defined by the condition (Def. 6).

(Def. 6) $\mathrm{C}_0(X) = \{f; f$ ranges over real maps of X: f is continuous $\wedge$ $\bigvee_{Y:\ \text{non empty subset of } X} (Y$ is compact $\wedge$ $\bigwedge_{A:\,\text{subset of } X} (A = \mathrm{support}\, f \ \Rightarrow\ \overline{A}$ is a subset of $Y))\}$.

The following propositions are true:

(26) For every non empty topological space X holds $\mathrm{C}_0(X)$ is a non empty non empty subset of RAlgebra (the carrier of X).

(27) Let X be a non empty topological space and W be a non empty subset of RAlgebra (the carrier of X). If $W = \mathrm{C}_0(X)$, then W is additively-linearly-closed.

(28) For every non empty topological space X holds $\mathrm{C}_0(X)$ is linearly closed.

Let X be a non empty topological space. Note that $\mathrm{C}_0(X)$ is non empty and linearly closed.

Let X be a non empty topological space. The functor $\mathrm{C}_0^{\mathrm{VS}}(X)$ yielding a real linear space is defined by:

(Def. 7) $\mathrm{C}_0^{\mathrm{VS}}(X) = \langle \mathrm{C}_0(X), \mathrm{Zero}(\mathrm{C}_0(X), \mathbb{R}_{\mathbb{R}}^{\text{the carrier of } X}), \mathrm{Add}(\mathrm{C}_0(X), \mathbb{R}_{\mathbb{R}}^{\text{the carrier of } X}), \mathrm{Mult}(\mathrm{C}_0(X), \mathbb{R}_{\mathbb{R}}^{\text{the carrier of } X})\rangle$.

The following two propositions are true:

(29) For every non empty topological space X holds $\mathrm{C}_0^{\mathrm{VS}}(X)$ is a subspace of $\mathbb{R}_{\mathbb{R}}^{\text{the carrier of } X}$.

(30) For every non empty topological space X and for every set x such that $x \in \mathrm{C}_0(X)$ holds $x \in$ BoundedFunctions (the carrier of X).

Let X be a non empty topological space. The functor $||\cdot||_{\mathrm{C}_0(X)}$ yielding a function from $\mathrm{C}_0(X)$ into $\mathbb{R}$ is defined by:

(Def. 8) $||\cdot||_{\mathrm{C}_0(X)} =$ BoundedFunctionsNorm (the carrier of X)$\restriction \mathrm{C}_0(X)$.

Let X be a non empty topological space. The functor $\mathrm{C}_0^{\mathrm{NS}}(X)$ yields a non empty normed structure and is defined as follows:

(Def. 9) $\mathrm{C}_0^{\mathrm{NS}}(X) = \langle \mathrm{C}_0(X), \mathrm{Zero}(\mathrm{C}_0(X), \mathbb{R}_{\mathbb{R}}^{\text{the carrier of } X}), \mathrm{Add}(\mathrm{C}_0(X), \mathbb{R}_{\mathbb{R}}^{\text{the carrier of } X}), \mathrm{Mult}(\mathrm{C}_0(X), \mathbb{R}_{\mathbb{R}}^{\text{the carrier of } X}), ||\cdot||_{\mathrm{C}_0(X)}\rangle$.

Let X be a non empty topological space. One can verify that $\mathrm{C}_0^{\mathrm{NS}}(X)$ is strict and non empty.

Next we state several propositions:

(31) For every non empty topological space X and for every set x such that $x \in \mathrm{C}_0(X)$ holds $x \in \mathrm{C}(X;\mathbb{R})$.

(32) For every non empty topological space X holds $0_{\mathrm{C}_0^{\mathrm{VS}}(X)} = X \longmapsto 0$.

(33) For every non empty topological space X holds $0_{\mathrm{C}_0^{\mathrm{NS}}(X)} = X \longmapsto 0$.

(34) Let a be a real number, X be a non empty topological space, and F, G be points of $\mathrm{C}_0^{\mathrm{NS}}(X)$. Then $\|F\| = 0$ iff $F = 0_{\mathrm{C}_0^{\mathrm{NS}}(X)}$ and $\|a \cdot F\| = |a| \cdot \|F\|$ and $\|F + G\| \leq \|F\| + \|G\|$.

(35) For every non empty topological space X holds $\mathrm{C}_0^{\mathrm{NS}}(X)$ is real normed space-like.

Let X be a non empty topological space. Note that $\mathrm{C}_0^{\mathrm{NS}}(X)$ is reflexive, discernible, real normed space-like, vector distributive, scalar distributive, scalar associative, scalar unital, Abelian, add-associative, right zeroed, and right complementable.

Next we state the proposition

(36) For every non empty topological space X holds $\mathrm{C}_0^{\mathrm{NS}}(X)$ is a real normed space.

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