Vector Functions and their Differentiation Formulas in 3-dimensional Euclidean Spaces

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Summary. In this article, we first extend several basic theorems of the operation of vector in 3-dimensional Euclidean spaces. Then three unit vectors: \(e_1, e_2, e_3\) and the definition of vector function in the same spaces are introduced. By dint of unit vector the main operation properties as well as the differentiation formulas of vector function are shown [12].

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The notation and terminology used in this paper have been introduced in the following papers: [7], [11], [2], [3], [4], [1], [5], [8], [9], [6], [10], and [13].

1. Preliminaries

For simplicity, we use the following convention: \(r, r_1, r_2, x, y, z, x_1, x_2, x_3, y_1, y_2, y_3\) are elements of \(\mathbb{R}\), \(p, q, p_1, p_2, p_3, q_1, q_2\) are elements of \(\mathbb{R}^3\), \(f_1, f_2, f_3, g_1, g_2, g_3, h_1, h_2, h_3\) are partial functions from \(\mathbb{R}\) to \(\mathbb{R}\), and \(t, t_0, t_1, t_2\) are real numbers.

Let \(x, y, z\) be real numbers. Then \([x, y, z]\) is an element of \(\mathbb{R}^3\).

One can prove the following proposition
(1) For every finite sequence $f$ of elements of $\mathbb{R}$ such that $\text{len } f = 3$ holds $f$ is an element of $\mathbb{R}^3$.

The element $e_1$ of $\mathbb{R}^3$ is defined by:

(Def. 1) $e_1 = [1, 0, 0]$.

The element $e_2$ of $\mathbb{R}^3$ is defined as follows:

(Def. 2) $e_2 = [0, 1, 0]$.

The element $e_3$ of $\mathbb{R}^3$ is defined as follows:

(Def. 3) $e_3 = [0, 0, 1]$.

Let us consider $p_1, p_2$. The functor $p_1 \times p_2$ yielding an element of $\mathbb{R}^3$ is defined as follows:

(Def. 4) $p_1 \times p_2 = [p_1(2) \cdot p_2(3) - p_1(3) \cdot p_2(2), p_1(3) \cdot p_2(1) - p_1(1) \cdot p_2(3), p_1(1) \cdot p_2(2) - p_1(2) \cdot p_2(1)]$.

Next we state the proposition

(2) If $p_1$ and $p_2$ are linearly dependent, then $p_1 \times p_2 = 0 e_1$.

2. Vector Functions in 3-dimensional Euclidean Spaces

We now state a number of propositions:

(3) $|e_1| = 1$.
(4) $|e_2| = 1$.
(5) $|e_3| = 1$.
(6) $e_1, e_2$ are orthogonal.
(7) $e_1, e_3$ are orthogonal.
(8) $e_2, e_3$ are orthogonal.
(9) $|(e_1, e_1)| = 1$.
(10) $|(e_2, e_2)| = 1$.
(11) $|(e_3, e_3)| = 1$.
(12) $|(e_1, [0, 0, 0])| = 0$.
(13) $|(e_2, [0, 0, 0])| = 0$.
(14) $|(e_3, [0, 0, 0])| = 0$.
(15) $e_1 \times e_2 = e_3$.
(16) $e_2 \times e_3 = e_1$.
(17) $e_3 \times e_1 = e_2$.
(18) $e_3 \times e_2 = -e_1$.
(19) $e_1 \times e_3 = -e_2$.
(20) $e_2 \times e_1 = -e_3$.
(21) $e_1 \times [0, 0, 0] = [0, 0, 0]$. 
(22) $e_2 \times [0, 0, 0] = [0, 0, 0]$.
(23) $e_3 \times [0, 0, 0] = [0, 0, 0]$.
(24) $r \cdot e_1 = [r, 0, 0]$.
(25) $r \cdot e_2 = [0, r, 0]$.
(26) $r \cdot e_3 = [0, 0, r]$.
(27) $1 \cdot e_1 = e_1$.
(28) $1 \cdot e_2 = e_2$.
(29) $1 \cdot e_3 = e_3$.
(30) $-e_1 = [-1, 0, 0]$.
(31) $-e_2 = [0, -1, 0]$.
(32) $-e_3 = [0, 0, -1]$.
(33) $0 \cdot e_1 = [0, 0, 0]$.
(34) $0 \cdot e_2 = [0, 0, 0]$.
(35) $0 \cdot e_3 = [0, 0, 0]$.
(36) $p = p(1) \cdot e_1 + p(2) \cdot e_2 + p(3) \cdot e_3$.
(37) $r \cdot p = r \cdot p(1) \cdot e_1 + r \cdot p(2) \cdot e_2 + r \cdot p(3) \cdot e_3$.
(38) $[x, y, z] = x \cdot e_1 + y \cdot e_2 + z \cdot e_3$.
(39) $r \cdot [x, y, z] = r \cdot x \cdot e_1 + r \cdot y \cdot e_2 + r \cdot z \cdot e_3$.
(40) $-p = -p(1) \cdot e_1 - p(2) \cdot e_2 - p(3) \cdot e_3$.
(41) $-[x, y, z] = -x \cdot e_1 - y \cdot e_2 - z \cdot e_3$.
(42) $p_1 + p_2 = (p_1(1) + p_2(1)) \cdot e_1 + (p_1(2) + p_2(2)) \cdot e_2 + (p_1(3) + p_2(3)) \cdot e_3$.
(43) $p_1 - p_2 = (p_1(1) - p_2(1)) \cdot e_1 + (p_1(2) - p_2(2)) \cdot e_2 + (p_1(3) - p_2(3)) \cdot e_3$.
(44) $[x_1, x_2, x_3] + [y_1, y_2, y_3] = (x_1 + y_1) \cdot e_1 + (x_2 + y_2) \cdot e_2 + (x_3 + y_3) \cdot e_3$.
(45) $[x_1, x_2, x_3] - [y_1, y_2, y_3] = (x_1 - y_1) \cdot e_1 + (x_2 - y_2) \cdot e_2 + (x_3 - y_3) \cdot e_3$.
(46) $p_1(1) \cdot e_1 + p_1(2) \cdot e_2 + p_1(3) \cdot e_3 = (p_1(1) + p_1(3)) \cdot e_1 + (p_2(2) + p_1(3)) \cdot e_2 + (p_2(3) + p_1(3)) \cdot e_3$ if and only if $p_2(1) \cdot e_1 + p_2(2) \cdot e_2 + p_2(3) \cdot e_3 = (p_1(1) - p_1(3)) \cdot e_1 + (p_1(2) - p_1(3)) \cdot e_2 + (p_1(3) - p_1(3)) \cdot e_3$.

Let $f_1, f_2, f_3$ be partial functions from $\mathbb{R}$ to $\mathbb{R}$. The functor $VFunc(f_1, f_2, f_3)$ yielding a function from $\mathbb{R}$ into $\mathbb{R}^3$ is defined as follows:

(Def. 5) For every $t$ holds $VFunc(f_1, f_2, f_3)(t) = [f_1(t), f_2(t), f_3(t)]$.

We now state a number of propositions:

(47) $(VFunc(f_1, f_2, f_3))(t) = f_1(t) \cdot e_1 + f_2(t) \cdot e_2 + f_3(t) \cdot e_3$.
(48) $p = (VFunc(f_1, f_2, f_3))(t)$ iff $p(1) = f_1(t)$ and $p(2) = f_2(t)$ and $p(3) = f_3(t)$.
(49) If $p = (VFunc(f_1, f_2, f_3))(t)$, then $\text{len} p = 3$ and $\text{dom} p = \text{Seg} 3$.
(50) If $p = (VFunc(f_1, f_2, f_3))(t_1)$ and $q = (VFunc(g_1, g_2, g_3))(t_2)$, then $p \cdot q = (f_1(t_1) \cdot g_1(t_2), f_2(t_1) \cdot g_2(t_2), f_3(t_1) \cdot g_3(t_2))$. 


(51) If \( p = (\text{VFunc}(f_1, f_2, f_3))(t) \), then \( r \cdot p = [r \cdot f_1(t), r \cdot f_2(t), r \cdot f_3(t)] \).

(52) If \( p = (\text{VFunc}(f_1, f_2, f_3))(t) \), then \( -p = [-f_1(t), -f_2(t), -f_3(t)] \).

(53) If \( p = (\text{VFunc}(f_1, f_2, f_3))(t) \), then \( (-p)(1) = -f_1(t) \) and \( (-p)(2) = -f_2(t) \) and \( (-p)(3) = -f_3(t) \).

(54) If \( p = (\text{VFunc}(f_1, f_2, f_3))(t) \), then \( \text{len}(p) = 3 \).

(55) If \( p = (\text{VFunc}(f_1, f_2, f_3))(t) \), then \( \text{len}(p) = \text{len}(p) \).

(56) If \( p = (\text{VFunc}(f_1, f_2, f_3))(t_1) \) and \( q = (\text{VFunc}(g_1, g_2, g_3))(t_2) \), then \( \text{len}(p + q) = 3 \).

(57) If \( p = (\text{VFunc}(f_1, f_2, f_3))(t_1) \) and \( q = (\text{VFunc}(g_1, g_2, g_3))(t_2) \), then \( p + q = [f_1(t_1) + g_1(t_2), f_2(t_1) + g_2(t_2), f_3(t_1) + g_3(t_2)] \).

(58) If \( p = (\text{VFunc}(f_1, f_2, f_3))(t_1) \) and \( q = (\text{VFunc}(g_1, g_2, g_3))(t_2) \) and \( p = q \), then \( f_1(t_1) = g_1(t_2) \) and \( f_2(t_1) = g_2(t_2) \) and \( f_3(t_1) = g_3(t_2) \).

(59) If \( f_1(t_1) = g_1(t_2) \) and \( f_2(t_1) = g_2(t_2) \) and \( f_3(t_1) = g_3(t_2) \), then \( \text{VFunc}(f_1, f_2, f_3))(t_1) = (VFunc(g_1, g_2, g_3))(t_2) \).

(60) If \( p = (\text{VFunc}(f_1, f_2, f_3))(t_1) \) and \( q = (\text{VFunc}(g_1, g_2, g_3))(t_2) \), then \( p + q = [f_1(t_1) - g_1(t_2), f_2(t_1) - g_2(t_2), f_3(t_1) - g_3(t_2)] \).

(61) If \( p = (\text{VFunc}(f_1, f_2, f_3))(t_1) \) and \( q = (\text{VFunc}(g_1, g_2, g_3))(t_2) \), then \( p - q = [f_1(t_1) - g_1(t_2), f_2(t_1) - g_2(t_2), f_3(t_1) - g_3(t_2)] \).

(62) If \( p = (\text{VFunc}(f_1, f_2, f_3))(t_1) \) and \( q = (\text{VFunc}(g_1, g_2, g_3))(t_2) \), then \( \text{len}(p) = \text{len}(q) \).

(63) If \( p = (\text{VFunc}(f_1, f_2, f_3))(t_1) \) and \( q = (\text{VFunc}(g_1, g_2, g_3))(t_2) \), then \( \text{len}(p) = 3 \).

(64) If \( p = (\text{VFunc}(f_1, f_2, f_3))(t_1) \) and \( q = (\text{VFunc}(g_1, g_2, g_3))(t_2) \), then \( |(p, q)| = f_1(t_1) \cdot g_1(t_2) + f_2(t_1) \cdot g_2(t_2) + f_3(t_1) \cdot g_3(t_2) \).

(65) If \( p = (\text{VFunc}(f_1, f_2, f_3))(t) \), then \( |(p, p)| = f_1(t)^2 + f_2(t)^2 + f_3(t)^2 \).

(66) If \( p = (\text{VFunc}(f_1, f_2, f_3))(t) \), then \( |p| = \sqrt{f_1(t)^2 + f_2(t)^2 + f_3(t)^2} \).

(67) If \( p = (\text{VFunc}(f_1, f_2, f_3))(t) \), then \( |r \cdot p| = |r| \cdot \sqrt{f_1(t)^2 + f_2(t)^2 + f_3(t)^2} \).

(68) If \( p = (\text{VFunc}(f_1, f_2, f_3))(t_1) \) and \( q = (\text{VFunc}(g_1, g_2, g_3))(t_2) \), then \( p \cdot q = [f_2(t_1) \cdot g_3(t_2) - f_3(t_1) \cdot g_2(t_2), f_3(t_1) \cdot g_1(t_2) - f_1(t_1) \cdot g_3(t_2), f_1(t_1) \cdot g_2(t_2) - f_2(t_1) \cdot g_1(t_2)] \).

(69) If \( p = (\text{VFunc}(f_1, f_2, f_3))(t) \), then \( r_1 \cdot p + r_2 \cdot p = (r_1 + r_2) \cdot [f_1(t), f_2(t), f_3(t)] \).

(70) If \( p = (\text{VFunc}(f_1, f_2, f_3))(t) \), then \( r_1 \cdot p - r_2 \cdot p = (r_1 - r_2) \cdot [f_1(t), f_2(t), f_3(t)] \).

(71) If \( p = (\text{VFunc}(f_1, f_2, f_3))(t_1) \) and \( q = (\text{VFunc}(g_1, g_2, g_3))(t_2) \), then \( |(r \cdot p, q)| = r \cdot (f_1(t_1) \cdot g_1(t_2) + f_2(t_1) \cdot g_2(t_2) + f_3(t_1) \cdot g_3(t_2)) \).

(72) If \( p = (\text{VFunc}(f_1, f_2, f_3))(t) \), then \( |(p, 0_{\mathbb{E}^3})| = 0 \).

(73) If \( p = (\text{VFunc}(f_1, f_2, f_3))(t_1) \) and \( q = (\text{VFunc}(g_1, g_2, g_3))(t_2) \), then \( |(p, q)| = -|(p, q)| \).
(74) If \( p = (\text{VFunc}(f_1, f_2, f_3))(t_1) \) and \( q = (\text{VFunc}(g_1, g_2, g_3))(t_2) \), then 
\[ |(-p, -q)| = |(p, q)|. \]
(75) If \( p_1 = (\text{VFunc}(f_1, f_2, f_3))(t_1) \) and \( p_2 = (\text{VFunc}(f_1, f_2, f_3))(t_2) \) and \( q = (\text{VFunc}(g_1, g_2, g_3))(t_2) \), then 
\[ |(p_1 - p_2, q)| = |(p_1, q)| - |(p_2, q)|. \]
(76) If \( p_1 = (\text{VFunc}(f_1, f_2, f_3))(t_1) \) and \( p_2 = (\text{VFunc}(f_1, f_2, f_3))(t_2) \) and \( q = (\text{VFunc}(g_1, g_2, g_3))(t_2) \), then 
\[ |(p_1 + p_2, q)| = |(p_1, q)| + |(p_2, q)|. \]
(77) If \( p_1 = (\text{VFunc}(f_1, f_2, f_3))(t_1) \) and \( p_2 = (\text{VFunc}(f_1, f_2, f_3))(t_2) \) and \( q = (\text{VFunc}(g_1, g_2, g_3))(t_2) \), then 
\[ |(r_1 \cdot p_1 + r_2 \cdot p_2, q)| = r_1 \cdot |(p_1, q)| + r_2 \cdot |(p_2, q)|. \]
(78) If \( p_1 = (\text{VFunc}(f_1, f_2, f_3))(t_1) \) and \( p_2 = (\text{VFunc}(f_1, f_2, f_3))(t_2) \) and \( q_1 = (\text{VFunc}(g_1, g_2, g_3))(t_1) \) and \( q_2 = (\text{VFunc}(g_1, g_2, g_3))(t_2) \), then 
\[ |(p_1 + p_2, q_1 + q_2)| = |(p_1, q_1)| + |(p_2, q_2)| + |(p_2, q_1)| + |(p_2, q_2)|. \]
(79) If \( p_1 = (\text{VFunc}(f_1, f_2, f_3))(t_1) \) and \( p_2 = (\text{VFunc}(f_1, f_2, f_3))(t_2) \) and \( q_1 = (\text{VFunc}(g_1, g_2, g_3))(t_1) \) and \( q_2 = (\text{VFunc}(g_1, g_2, g_3))(t_2) \), then 
\[ |(p_1 - p_2, q_1 - q_2)| = |(p_1, q_1)| - |(p_2, q_2)| - |(p_1, q_1)| + |(p_2, q_2)|. \]
(80) For every \( p \) such that \( p = (\text{VFunc}(f_1, f_2, f_3))(t) \) holds \( |(p, p)| = 0 \) iff \( p = 0 \).
(81) For every \( p \) such that \( p = (\text{VFunc}(f_1, f_2, f_3))(t) \) holds \( |p| = 0 \) iff \( p = 0 \).
(82) If \( p = (\text{VFunc}(f_1, f_2, f_3))(t) \) and \( q = (\text{VFunc}(g_1, g_2, g_3))(t) \), then 
\[ |(p - q, p - q)| = |(p, p)| - 2 \cdot |(p, q)| + |(q, q)|. \]
(83) If \( p = (\text{VFunc}(f_1, f_2, f_3))(t) \) and \( q = (\text{VFunc}(g_1, g_2, g_3))(t) \), then 
\[ |(p + q, p + q)| = |(p, p)| + 2 \cdot |(p, q)| + |(q, q)|. \]
(84) If \( p = (\text{VFunc}(f_1, f_2, f_3))(t) \) and \( q = (\text{VFunc}(g_1, g_2, g_3))(t) \), then 
\[ (r \cdot p) \times q = r \cdot (p \times q) \quad \text{and} \quad (r \cdot p) \times q = p \times (r \cdot q). \]
(85) If \( p_1 = (\text{VFunc}(f_1, f_2, f_3))(t_1) \) and \( p_2 = (\text{VFunc}(f_1, f_2, f_3))(t_2) \) and \( q = (\text{VFunc}(g_1, g_2, g_3))(t) \), then 
\[ p_1 \times (p_2 + q) = p_1 \times p_2 + p_1 \times q. \]
(86) If \( p_1 = (\text{VFunc}(f_1, f_2, f_3))(t_1) \) and \( p_2 = (\text{VFunc}(f_1, f_2, f_3))(t_2) \) and \( q = (\text{VFunc}(g_1, g_2, g_3))(t) \), then 
\[ (p_1 + p_2) \times q = p_1 \times q + p_2 \times q. \]

Let us consider \( p_1, p_2, p_3 \). The functor \( \langle p_1, p_2, p_3 \rangle \) yields a real number and is defined as follows:

(Def. 6) \[ \langle |p_1, p_2, p_3| \rangle = \langle |p_1, p_2 \times p_3| \rangle. \]

Next we state several propositions:

(87) If \( p_1 = (\text{VFunc}(f_1, f_2, f_3))(t_1) \) and \( p_2 = (\text{VFunc}(f_1, f_2, f_3))(t_2) \), then 
\[ \langle |p_1, p_2| \rangle = 0. \]
(88) If \( p_1 = (\text{VFunc}(f_1, f_2, f_3))(t_1) \) and \( p_2 = (\text{VFunc}(f_1, f_2, f_3))(t_2) \), then 
\[ \langle |p_2, p_1| \rangle = 0. \]
(89) If \( p_1 = (\text{VFunc}(f_1, f_2, f_3))(t_1) \) and \( p_2 = (\text{VFunc}(f_1, f_2, f_3))(t_2) \), then 
\[ \langle |p_1, p_2| \rangle = 0. \]
(90) If \( p_1 = (\text{VFunc}(f_1, f_2, f_3))(t_1) \) and \( p_2 = (\text{VFunc}(f_1, f_2, f_3))(t_2) \) and \( q = (\text{VFunc}(g_1, g_2, g_3))(t) \), then 
\[ \langle |p_1, p_2, q| \rangle = \langle |p_2, q, p_1| \rangle. \]
Let \( f_1, f_2, f_3 \) be partial functions from \( \mathbb{R} \) to \( \mathbb{R} \) and let \( t_0 \) be a real number. The functor \( \text{VFunct}(f_1, f_2, f_3, t_0) \) yielding an element of \( \mathbb{R}^3 \) is defined as follows:

**Def. 7** \( \text{VFunct}(f_1, f_2, f_3, t_0) = [f_1'(t_0), f_2'(t_0), f_3'(t_0)]. \)

Next we state a number of propositions:

(91) Suppose \( f_1, f_2 \) and \( f_3 \) are differentiable in \( t_0 \) and \( f_2 \) is differentiable in \( t_0 \) and \( f_3 \) is differentiable in \( t_0 \). Then \( \text{VFunct}(f_1, f_2, f_3, t_0) = f_1'(t_0) \cdot e_1 + f_2'(t_0) \cdot e_2 + f_3'(t_0) \cdot e_3. \)

(92) Suppose \( f_1, f_2 \) and \( f_3 \) are differentiable in \( t_0 \) and \( q = (VFunc\langle f_1, f_2, f_3 \rangle(t_0), then \|p_1, p_2, q\| = \|p_1 \times p_2, q\|). \)

(93) Suppose that \( f_1 \) is differentiable in \( t_0 \) and \( f_2 \) is differentiable in \( t_0 \) and \( f_3 \) is differentiable in \( t_0 \). Then \( \text{VFunct}(f_1, f_2, f_3, t_0) = f_1'(t_0) \cdot e_1 + f_2'(t_0) \cdot e_2 + f_3'(t_0) \cdot e_3. \)

(94) Suppose that

- (i) \( f_1 \) is differentiable in \( t_0 \),
- (ii) \( f_2 \) is differentiable in \( t_0 \),
- (iii) \( f_3 \) is differentiable in \( t_0 \),
- (iv) \( g_1 \) is differentiable in \( t_0 \),
- (v) \( g_2 \) is differentiable in \( t_0 \), and
- (vi) \( g_3 \) is differentiable in \( t_0 \).

Then \( \text{VFunct}(f_1 + g_1, f_2 + g_2, f_3 + g_3, t_0) = \text{VFunct}(f_1, f_2, f_3, t_0) + \text{VFunct}(g_1, g_2, g_3, t_0). \)

(95) Suppose that

- (i) \( f_1 \) is differentiable in \( t_0 \),
- (ii) \( f_2 \) is differentiable in \( t_0 \),
- (iii) \( f_3 \) is differentiable in \( t_0 \),
- (iv) \( g_1 \) is differentiable in \( t_0 \),
- (v) \( g_2 \) is differentiable in \( t_0 \), and
- (vi) \( g_3 \) is differentiable in \( t_0 \).

Then \( \text{VFunct}(f_1 - g_1, f_2 - g_2, f_3 - g_3, t_0) = \text{VFunct}(f_1, f_2, f_3, t_0) - \text{VFunct}(g_1, g_2, g_3, t_0). \)

(96) If \( f_1 \) is differentiable in \( t_0 \) and \( f_2 \) is differentiable in \( t_0 \) and \( f_3 \) is differentiable in \( t_0 \), then \( \text{VFunct}(r \cdot f_1, r \cdot f_2, r \cdot f_3, t_0) = r \cdot \text{VFunct}(f_1, f_2, f_3, t_0). \)

(97) Suppose that

- (i) \( f_1 \) is differentiable in \( t_0 \),
- (ii) \( f_2 \) is differentiable in \( t_0 \), and
- (iii) \( f_3 \) is differentiable in \( t_0 \),
(iv) \( g_1 \) is differentiable in \( t_0 \),
(v) \( g_2 \) is differentiable in \( t_0 \), and
(vi) \( g_3 \) is differentiable in \( t_0 \).

Then \( \text{VFunctdiff}(f_1, g_1, f_2, g_2, f_3, g_3, t_0) = [g_1(t_0) \cdot f_1'(t_0), g_2(t_0) \cdot f_2'(t_0), g_3(t_0) \cdot f_3'(t_0)] + [f_1(t_0) \cdot g_1'(t_0), f_2(t_0) \cdot g_2'(t_0), f_3(t_0) \cdot g_3'(t_0)]. \)

(98) Suppose that
(i) \( f_1 \) is differentiable in \( t_0 \),
(ii) \( f_2 \) is differentiable in \( t_0 \),
(iii) \( f_3 \) is differentiable in \( t_0 \),
(iv) \( g_1 \) is differentiable in \( f_1(t_0) \),
(v) \( g_2 \) is differentiable in \( f_2(t_0) \), and
(vi) \( g_3 \) is differentiable in \( f_3(t_0) \).

Then \( \text{VFunctdiff}(g_1 \cdot f_1, g_2 \cdot f_2, g_3 \cdot f_3, t_0) = [g_1'(f_1(t_0)) \cdot f_1'(t_0), g_2'(f_2(t_0)) \cdot f_2'(t_0), g_3'(f_3(t_0)) \cdot f_3'(t_0)]. \)

(99) Suppose that \( f_1 \) is differentiable in \( t_0 \) and \( f_2 \) is differentiable in \( t_0 \) and \( f_3 \) is differentiable in \( t_0 \) and \( g_1 \) is differentiable in \( t_0 \) and \( g_2 \) is differentiable in \( t_0 \) and \( g_3 \) is differentiable in \( t_0 \) and \( g_1(t_0) \neq 0 \) and \( g_2(t_0) \neq 0 \) and \( g_3(t_0) \neq 0 \). Then \( \text{VFunctdiff}(f_1, f_2, f_3, t_0) = g_1(t_0) \cdot g_2(t_0) \cdot g_3(t_0) \cdot \frac{f_1'(t_0), f_2'(t_0), f_3'(t_0) - g_1'(t_0) \cdot f_1(t_0)}{g_1(t_0)^2, g_2(t_0)^2, g_3(t_0)^2}. \)

(100) Suppose \( f_1 \) is differentiable in \( t_0 \) and \( f_2 \) is differentiable in \( t_0 \) and \( f_3 \) is differentiable in \( t_0 \) and \( f_1(t_0) \neq 0 \) and \( f_2(t_0) \neq 0 \) and \( f_3(t_0) \neq 0 \). Then \( \text{VFunctdiff}(\frac{1}{f_1}, \frac{1}{f_2}, \frac{1}{f_3}, t_0) = -[\frac{f_1'(t_0), f_2'(t_0), f_3'(t_0)}{f_1(t_0)^2, f_2(t_0)^2, f_3(t_0)^2}]. \)

(101) Suppose \( f_1 \) is differentiable in \( t_0 \) and \( f_2 \) is differentiable in \( t_0 \) and \( f_3 \) is differentiable in \( t_0 \). Then \( \text{VFunctdiff}(r \cdot f_1, r \cdot f_2, r \cdot f_3, t_0) = r \cdot f_1'(t_0) \cdot e_1 + r \cdot f_2'(t_0) \cdot e_2 + r \cdot f_3'(t_0) \cdot e_3. \)

(102) Suppose that
(i) \( f_1 \) is differentiable in \( t_0 \),
(ii) \( f_2 \) is differentiable in \( t_0 \),
(iii) \( f_3 \) is differentiable in \( t_0 \),
(iv) \( g_1 \) is differentiable in \( t_0 \),
(v) \( g_2 \) is differentiable in \( t_0 \), and
(vi) \( g_3 \) is differentiable in \( t_0 \).

Then \( \text{VFunctdiff}(r \cdot f_1 + g_1, r \cdot (f_2 + g_2), r \cdot (f_3 + g_3), t_0) = r \cdot \text{VFunctdiff}(f_1, f_2, f_3, t_0) + r \cdot \text{VFunctdiff}(g_1, g_2, g_3, t_0). \)

(103) Suppose that
(i) \( f_1 \) is differentiable in \( t_0 \),
(ii) \( f_2 \) is differentiable in \( t_0 \),
(iii) \( f_3 \) is differentiable in \( t_0 \),
(iv) \( g_1 \) is differentiable in \( t_0 \),
(v) \( g_2 \) is differentiable in \( t_0 \), and
(vi) \( g_3 \) is differentiable in \( t_0 \).

Then \( \text{VFuncdiff}(r (f_1 - g_1), r (f_2 - g_2), r (f_3 - g_3), t_0) = r \cdot \text{VFuncdiff}(f_1, f_2, f_3, t_0) - r \cdot \text{VFuncdiff}(g_1, g_2, g_3, t_0) \).

(104) Suppose that

(i) \( f_1 \) is differentiable in \( t_0 \),
(ii) \( f_2 \) is differentiable in \( t_0 \),
(iii) \( f_3 \) is differentiable in \( t_0 \),
(iv) \( g_1 \) is differentiable in \( t_0 \),
(v) \( g_2 \) is differentiable in \( t_0 \), and
(vi) \( g_3 \) is differentiable in \( t_0 \).

Then
\[
\text{VFuncdiff}(r f_1, r f_2, r f_3, t_0) = r \cdot \text{VFuncdiff}(f_1, f_2, f_3, t_0) + \frac{1}{r} \cdot \left[ f_1'(t_0) \cdot g_1(t_0) - g_1'(t_0) \cdot f_1(t_0) \right] - \frac{1}{r} \cdot \text{VFuncdiff}(g_1, g_2, g_3, t_0).
\]

(105) Suppose that

(i) \( f_1 \) is differentiable in \( t_0 \),
(ii) \( f_2 \) is differentiable in \( t_0 \),
(iii) \( f_3 \) is differentiable in \( t_0 \),
(iv) \( g_1 \) is differentiable in \( f_1(t_0) \),
(v) \( g_2 \) is differentiable in \( f_2(t_0) \), and
(vi) \( g_3 \) is differentiable in \( f_3(t_0) \).

Then
\[
\text{VFuncdiff}(r f_1, r f_2, r f_3, t_0) = r \cdot \text{VFuncdiff}(f_1, f_2, f_3, t_0) + \frac{1}{r} \cdot \left[ g_1'(t_0) \cdot f_1(t_0) - f_1'(t_0) \cdot g_1(t_0) \right] - \frac{1}{r} \cdot \text{VFuncdiff}(g_1, g_2, g_3, t_0).
\]

(106) Suppose that \( f_1 \) is differentiable in \( t_0 \) and \( f_2 \) is differentiable in \( t_0 \) and \( f_3 \) is differentiable in \( t_0 \) and \( g_1 \) is differentiable in \( t_0 \) and \( g_2 \) is differentiable in \( t_0 \) and \( g_3 \) is differentiable in \( t_0 \) and \( g_1(t_0) \neq 0 \) and \( g_2(t_0) \neq 0 \) and \( g_3(t_0) \neq 0 \). Then
\[
\text{VFuncdiff}(r f_1, r f_2, r f_3, t_0) = r \cdot \left[ \frac{f_1'(t_0) \cdot g_1(t_0) - g_1'(t_0) \cdot f_1(t_0)}{g_1(t_0)^2} \right] - \frac{1}{r} \cdot \text{VFuncdiff}(g_1, g_2, g_3, t_0).
\]

(107) Suppose that \( f_1 \) is differentiable in \( t_0 \) and \( f_2 \) is differentiable in \( t_0 \) and \( f_3 \) is differentiable in \( t_0 \) and \( f_1(t_0) \neq 0 \) and \( f_2(t_0) \neq 0 \) and \( f_3(t_0) \neq 0 \) and \( r \neq 0 \). Then
\[
\text{VFuncdiff}(r f_1, r f_2, r f_3, t_0) = - \frac{1}{r} \cdot \left[ \frac{f_1'(t_0)}{f_1(t_0)^2} \cdot \frac{f_2'(t_0)}{f_2(t_0)^2} \cdot \frac{f_3'(t_0)}{f_3(t_0)^2} \right].
\]

(108) Suppose that

(i) \( f_1 \) is differentiable in \( t_0 \),
(ii) \( f_2 \) is differentiable in \( t_0 \),
(iii) \( f_3 \) is differentiable in \( t_0 \),
(iv) \( g_1 \) is differentiable in \( t_0 \),
(v) \( g_2 \) is differentiable in \( t_0 \), and
(vi) \( g_3 \) is differentiable in \( t_0 \).

Then
\[
\text{VFuncdiff}(f_2 g_3 - f_3 g_2, f_3 g_1 - f_1 g_3, f_1 g_2 - f_2 g_1, t_0) = \left[ f_2(t_0) \cdot g_3'(t_0) - f_3(t_0) \cdot g_2'(t_0), f_3(t_0) \cdot g_1(t_0) - f_1(t_0) \cdot g_3(t_0), f_1(t_0) \cdot g_2(t_0) - f_2(t_0) \cdot g_1(t_0) \right].
\]
(109) Suppose that \( f_1 \) is differentiable in \( t_0 \) and \( f_2 \) is differentiable in \( t_0 \) and \( f_3 \) is differentiable in \( t_0 \) and \( g_1 \) is differentiable in \( t_0 \) and \( g_2 \) is differentiable in \( t_0 \) and \( g_3 \) is differentiable in \( t_0 \) and \( h_1 \) is differentiable in \( t_0 \) and \( h_2 \) is differentiable in \( t_0 \) and \( h_3 \) is differentiable in \( t_0 \). Then \( \text{VFucdiff} \{ h_1 (f_2 g_3 - f_3 g_2) \cdot h_2 (f_3 g_1 - f_1 g_3), h_3 (f_1 g_2 - f_2 g_1), t_0 \} = [h_1' (t_0) \cdot (f_2 (t_0) \cdot g_3 (t_0) - f_3 (t_0) \cdot g_2 (t_0)), h_2' (t_0) \cdot (f_3 (t_0) \cdot g_1 (t_0) - f_1 (t_0) \cdot g_3 (t_0)), h_3' (t_0) \cdot (f_1 (t_0) \cdot g_2 (t_0) - f_2 (t_0) \cdot g_1 (t_0)) + [h_1 (t_0) \cdot (f_2' (t_0) \cdot g_3 (t_0) - f_3' (t_0) \cdot g_2 (t_0)), h_2 (t_0) \cdot (f_3' (t_0) \cdot g_1 (t_0) - f_1' (t_0) \cdot g_3 (t_0)), h_3 (t_0) \cdot (f_1' (t_0) \cdot g_2 (t_0) - f_2' (t_0) \cdot g_1 (t_0))] + [h_1 (t_0) \cdot (f_2 (t_0) \cdot g_3 (t_0) - f_3 (t_0) \cdot g_2 (t_0)), h_2 (t_0) \cdot (f_3 (t_0) \cdot g_1 (t_0) - f_1 (t_0) \cdot g_3 (t_0)), h_3 (t_0) \cdot (f_1 (t_0) \cdot g_2 (t_0) - f_2 (t_0) \cdot g_1 (t_0))].

(110) Suppose that \( f_1 \) is differentiable in \( t_0 \) and \( f_2 \) is differentiable in \( t_0 \) and \( f_3 \) is differentiable in \( t_0 \) and \( g_1 \) is differentiable in \( t_0 \) and \( g_2 \) is differentiable in \( t_0 \) and \( g_3 \) is differentiable in \( t_0 \) and \( h_1 \) is differentiable in \( t_0 \) and \( h_2 \) is differentiable in \( t_0 \). Then \( \text{VFucdiff} \{ h_2 (f_2 g_3 - h_3 f_3 g_2) \cdot h_3 f_3 g_1 - h_1 f_1 g_2 - h_2 f_2 g_1, t_0 \} = [h_2 (t_0) \cdot (f_2 (t_0) \cdot g_3 (t_0) - h_3 (t_0) \cdot f_3 (t_0) \cdot g_2 (t_0) - h_3 (t_0) \cdot f_3 (t_0) \cdot g_1 (t_0) - h_1 (t_0) \cdot f_1 (t_0) \cdot g_3 (t_0), h_1 (t_0) \cdot f_1 (t_0) \cdot g_2 (t_0) - h_2 (t_0) \cdot f_2 (t_0) \cdot g_1 (t_0))] + [h_2 (t_0) \cdot (f_2 (t_0) \cdot g_3 (t_0) - h_3 (t_0) \cdot f_3 (t_0) \cdot g_2 (t_0) - h_3 (t_0) \cdot f_3 (t_0) \cdot g_1 (t_0) - h_1 (t_0) \cdot f_1 (t_0) \cdot g_3 (t_0), h_1 (t_0) \cdot f_1 (t_0) \cdot g_2 (t_0) - h_2 (t_0) \cdot f_2 (t_0) \cdot g_1 (t_0))].

(111) Suppose that \( f_1 \) is differentiable in \( t_0 \) and \( f_2 \) is differentiable in \( t_0 \) and \( f_3 \) is differentiable in \( t_0 \) and \( g_1 \) is differentiable in \( t_0 \) and \( g_2 \) is differentiable in \( t_0 \) and \( g_3 \) is differentiable in \( t_0 \) and \( h_1 \) is differentiable in \( t_0 \) and \( h_2 \) is differentiable in \( t_0 \). Then \( \text{VFucdiff} \{ h_2 (f_1 g_2 - f_2 g_1) - h_3 (f_3 g_1 - f_1 g_3), h_3 (f_2 g_3 - f_3 g_2) - h_1 (f_1 g_2 - f_2 g_1), h_1 (f_3 g_1 - f_1 g_3) - h_2 (f_2 g_3 - f_3 g_2), t_0 \} = [h_2 (t_0) \cdot (f_1 (t_0) \cdot g_2 (t_0) - f_2 (t_0) \cdot g_1 (t_0)) - h_3 (t_0) \cdot (f_3 (t_0) \cdot g_1 (t_0) - f_1 (t_0) \cdot g_3 (t_0)), h_3 (t_0) \cdot (f_2 (t_0) \cdot g_3 (t_0) - f_3 (t_0) \cdot g_2 (t_0)), h_1 (t_0) \cdot (f_1 (t_0) \cdot g_2 (t_0) - f_2 (t_0) \cdot g_1 (t_0)) - h_2 (t_0) \cdot (f_2 (t_0) \cdot g_3 (t_0) - f_3 (t_0) \cdot g_2 (t_0)) + [h_2 (t_0) \cdot (f_1 (t_0) \cdot g_2 (t_0) - f_2 (t_0) \cdot g_1 (t_0)) - h_3 (t_0) \cdot (f_3 (t_0) \cdot g_1 (t_0) - f_1 (t_0) \cdot g_3 (t_0)), h_3 (t_0) \cdot (f_2 (t_0) \cdot g_3 (t_0) - f_3 (t_0) \cdot g_2 (t_0)), h_1 (t_0) \cdot (f_1 (t_0) \cdot g_2 (t_0) - f_2 (t_0) \cdot g_1 (t_0))] + [h_2 (t_0) \cdot (f_1 (t_0) \cdot g_2 (t_0) - f_2 (t_0) \cdot g_1 (t_0)) - h_3 (t_0) \cdot (f_3 (t_0) \cdot g_1 (t_0) - f_1 (t_0) \cdot g_3 (t_0)), h_3 (t_0) \cdot (f_2 (t_0) \cdot g_3 (t_0) - f_3 (t_0) \cdot g_2 (t_0)), h_1 (t_0) \cdot (f_1 (t_0) \cdot g_2 (t_0) - f_2 (t_0) \cdot g_1 (t_0)) - h_2 (t_0) \cdot (f_2 (t_0) \cdot g_3 (t_0) - f_3 (t_0) \cdot g_2 (t_0))].

References


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