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Small Inductive Dimension of Topological Spaces

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Summary. We present the concept and basic properties of the Menger-Urysohn small inductive dimension of topological spaces according to the books [7]. Namely, the paper includes the formalization of main theorems from Sections 1.1 and 1.2.

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The terminology and notation used here are introduced in the following articles: [17], [8], [15], [5], [16], [6], [18], [14], [1], [2], [3], [13], [11], [9], [12], [19], [20], [10], and [4].

1. Preliminaries

For simplicity, we adopt the following rules: T, T_1, T_2 denote topological spaces, A, B denote subsets of T, F denotes a subset of $T\restriction A$, G, G_1, G_2 denote families of subsets of T, U, W denote open subsets of $T\restriction A$, p denotes a point of $T\restriction A$, n denotes a natural number, and I denotes an integer.

One can prove the following propositions:

(1) $\mathrm{Fr}(B \cap A) \subseteq \mathrm{Fr}\, B \cap A$.

(2) T is a T_4 space if and only if for all closed subsets A, B of T such that A misses B there exist open subsets U, W of T such that $A \subseteq U$ and $B \subseteq W$ and $\overline{U}$ misses $\overline{W}$.

Let us consider T. The sequence of ind of T yields a sequence of subsets of $2^{\text{the carrier of } T}$ and is defined by the conditions (Def. 1).

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(Def. 1)(i) (The sequence of ind of T)$(0) = \{\emptyset_T\}$, and

(ii) $A \in$ (the sequence of ind of T)$(n+1)$ iff $A \in$ (the sequence of ind of T)(n) or for all p, U such that $p \in U$ there exists W such that $p \in W$ and $W \subseteq U$ and $\operatorname{Fr} W \in$ (the sequence of ind of T)(n).

Let us consider T. Note that the sequence of ind of T is ascending.

We now state the proposition

(3) For every F such that $F = B$ holds $F \in$ (the sequence of ind of $T \restriction A$)(n) iff $B \in$ (the sequence of ind of T)(n).

Let us consider T, A. We say that A has finite small inductive dimension if and only if:

(Def. 2) There exists n such that $A \in$ (the sequence of ind of T)(n).

Let us consider T, A. We introduce A is finite-ind as a synonym of A has finite small inductive dimension.

Let us consider T, G. We say that G has finite small inductive dimension if and only if:

(Def. 3) There exists n such that $G \subseteq$ (the sequence of ind of T)(n).

Let us consider T, G. We introduce G is finite-ind as a synonym of G has finite small inductive dimension.

The following proposition is true

(4) If $A \in G$ and G is finite-ind, then A is finite-ind.

Let us consider T. One can check the following observations:

* every subset of T which is finite is also finite-ind,
* there exists a subset of T which is finite-ind,
* every family of subsets of T which is empty is also finite-ind, and
* there exists a family of subsets of T which is non empty and finite-ind.

Let T be a non empty topological space. One can check that there exists a subset of T which is non empty and finite-ind.

Let us consider T. We say that T has finite small inductive dimension if and only if:

(Def. 4) Ω_T has finite small inductive dimension.

Let us consider T. We introduce T is finite-ind as a synonym of T has finite small inductive dimension.

One can verify that every topological space which is empty is also finite-ind.

Let X be a set. Note that $\{X\}_{\rm top}$ is finite-ind.

One can check that there exists a topological space which is non empty and finite-ind.

In the sequel A_1 is a finite-ind subset of T and T_3 is a finite-ind topological space.

2. Small Inductive Dimension

Let us consider T and let us consider A. Let us assume that A is finite-ind. The functor $\operatorname{ind} A$ yields an integer and is defined as follows:

(Def. 5) $A \in$ (the sequence of ind of T)($\operatorname{ind} A + 1$) and $A \notin$ (the sequence of ind of T)($\operatorname{ind} A$).

We now state two propositions:

(5) $-1 \leq \operatorname{ind} A_1$.

(6) $\operatorname{ind} A_1 = -1$ iff A_1 is empty.

Let T be a non empty topological space and let A be a non empty finite-ind subset of T. Observe that $\operatorname{ind} A$ is natural.

The following three propositions are true:

(7) $\operatorname{ind} A_1 \leq n - 1$ iff $A_1 \in$ (the sequence of ind of T)(n).

(8) For every finite subset A of T holds $\operatorname{ind} A < \overline{\overline{A}}$.

(9) $\operatorname{ind} A_1 \leq n$ if and only if for every point p of $T \restriction A_1$ and for every open subset U of $T \restriction A_1$ such that $p \in U$ there exists an open subset W of $T \restriction A_1$ such that $p \in W$ and $W \subseteq U$ and $\operatorname{Fr} W$ is finite-ind and $\operatorname{ind} \operatorname{Fr} W \leq n - 1$.

Let us consider T and let us consider G. Let us assume that G is finite-ind. The functor $\operatorname{ind} G$ yielding an integer is defined by the conditions (Def. 6).

(Def. 6)(i) $G \subseteq$ (the sequence of ind of T)($\operatorname{ind} G + 1$),

(ii) $-1 \leq \operatorname{ind} G$, and

(iii) for every integer i such that $-1 \leq i$ and $G \subseteq$ (the sequence of ind of T)($i + 1$) holds $\operatorname{ind} G \leq i$.

The following propositions are true:

(10) $\operatorname{ind} G = -1$ and G is finite-ind iff $G \subseteq \{\emptyset_T\}$.

(11) G is finite-ind and $\operatorname{ind} G \leq I$ iff $-1 \leq I$ and for every A such that $A \in G$ holds A is finite-ind and $\operatorname{ind} A \leq I$.

(12) If G_1 is finite-ind and $G_2 \subseteq G_1$, then G_2 is finite-ind and $\operatorname{ind} G_2 \leq \operatorname{ind} G_1$.

Let us consider T and let G_1, G_2 be finite-ind families of subsets of T. Observe that $G_1 \cup G_2$ is finite-ind.

The following proposition is true

(13) If G is finite-ind and G_1 is finite-ind and $\operatorname{ind} G \leq I$ and $\operatorname{ind} G_1 \leq I$, then $\operatorname{ind}(G \cup G_1) \leq I$.

Let us consider T. The functor $\operatorname{ind} T$ yields an integer and is defined as follows:

(Def. 7) $\operatorname{ind} T = \operatorname{ind}(\Omega_T)$.

Let T be a non empty finite-ind topological space. One can verify that $\operatorname{ind} T$ is natural.

The following three propositions are true:

(14) For every non empty set X holds $\operatorname{ind}(\{X\}_{\rm top}) = 0$.

(15) Given n such that let p be a point of T and U be an open subset of T. Suppose $p \in U$. Then there exists an open subset W of T such that $p \in W$ and $W \subseteq U$ and $\operatorname{Fr} W$ is finite-ind and $\operatorname{ind} \operatorname{Fr} W \leq n - 1$. Then T is finite-ind.

(16) $\operatorname{ind} T_3 \leq n$ if and only if for every point p of T_3 and for every open subset U of T_3 such that $p \in U$ there exists an open subset W of T_3 such that $p \in W$ and $W \subseteq U$ and $\operatorname{Fr} W$ is finite-ind and $\operatorname{ind} \operatorname{Fr} W \leq n - 1$.

3. Monotonicity of the Small Inductive Dimension

Let us consider T_3. Observe that every subset of T_3 is finite-ind.

Let us consider T, A_1. Note that $T \restriction A_1$ is finite-ind.

One can prove the following propositions:

(17) $\operatorname{ind}(T \restriction A_1) = \operatorname{ind} A_1$.

(18) If $T \restriction A$ is finite-ind, then A is finite-ind.

(19) If $A \subseteq A_1$, then A is finite-ind and $\operatorname{ind} A \leq \operatorname{ind} A_1$.

(20) For every subset A of T_3 holds $\operatorname{ind} A \leq \operatorname{ind} T_3$.

(21) If $F = B$ and B is finite-ind, then F is finite-ind and $\operatorname{ind} F = \operatorname{ind} B$.

(22) If $F = B$ and F is finite-ind, then B is finite-ind and $\operatorname{ind} F = \operatorname{ind} B$.

(23) Let T be a non empty topological space. Suppose T is a T_3 space. Then T is finite-ind and $\operatorname{ind} T \leq n$ if and only if for every closed subset A of T and for every point p of T such that $p \notin A$ there exists a subset L of T such that L separates $\{p\}$, A and L is finite-ind and $\operatorname{ind} L \leq n - 1$.

(24) If T_1 and T_2 are homeomorphic, then T_1 is finite-ind iff T_2 is finite-ind.

(25) If T_1 and T_2 are homeomorphic and T_1 is finite-ind, then $\operatorname{ind} T_1 = \operatorname{ind} T_2$.

(26) Let A_2 be a subset of T_1 and A_3 be a subset of T_2. Suppose A_2 and A_3 are homeomorphic. Then A_2 is finite-ind if and only if A_3 is finite-ind.

(27) Let A_2 be a subset of T_1 and A_3 be a subset of T_2. If A_2 and A_3 are homeomorphic and A_2 is finite-ind, then $\operatorname{ind} A_2 = \operatorname{ind} A_3$.

(28) If $T_1 \times T_2$ is finite-ind, then $T_2 \times T_1$ is finite-ind and $\operatorname{ind}(T_1 \times T_2) = \operatorname{ind}(T_2 \times T_1)$.

(29) For every family G_3 of subsets of $T \restriction A$ such that G_3 is finite-ind and $G_3 = G$ holds G is finite-ind and $\operatorname{ind} G = \operatorname{ind} G_3$.

(30) For every family G_3 of subsets of $T \restriction A$ such that G is finite-ind and $G_3 = G$ holds G_3 is finite-ind and $\operatorname{ind} G = \operatorname{ind} G_3$.

4. Basic Properties 0-dimensional Topological Spaces

Next we state several propositions:

(31) T is finite-ind and $\operatorname{ind} T \leq n$ if and only if there exists a basis B_1 of T such that for every A such that $A \in B_1$ holds $\operatorname{Fr} A$ is finite-ind and $\operatorname{ind} \operatorname{Fr} A \leq n - 1$.

(32) Let given T. Suppose that

(i) T is a T_1 space, and

(ii) for all closed subsets A, B of T such that A misses B there exist closed subsets A', B' of T such that A' misses B' and $A' \cup B' = \Omega_T$ and $A \subseteq A'$ and $B \subseteq B'$.

Then T is finite-ind and $\operatorname{ind} T \leq 0$.

(33) Let X be a set and f be a sequence of subsets of X. Then there exists a sequence g of subsets of X such that

(i) $\bigcup \operatorname{rng} f = \bigcup \operatorname{rng} g$,

(ii) for all natural numbers i, j such that $i \neq j$ holds $g(i)$ misses $g(j)$, and

(iii) for every n there exists a finite family f_1 of subsets of X such that $f_1 = \{f(i); i$ ranges over elements of $\mathbb{N}$: $i < n\}$ and $g(n) = f(n) \setminus \bigcup f_1$.

(34) Let given T. Suppose T is finite-ind and $\operatorname{ind} T \leq 0$ and T is Lindelöf. Let A, B be closed subsets of T. Suppose A misses B. Then there exist closed subsets A', B' of T such that A' misses B' and $A' \cup B' = \Omega_T$ and $A \subseteq A'$ and $B \subseteq B'$.

(35) Let given T. Suppose T is a T_1 space and Lindelöf. Then T is finite-ind and $\operatorname{ind} T \leq 0$ if and only if for all closed subsets A, B of T such that A misses B holds $\emptyset_T$ separates A, B.

(36) Let given T. Suppose that

(i) T is a T_4 space, a T_1 space, and Lindelöf, and

(ii) there exists a family F of subsets of T such that F is closed, a cover of T, countable, and finite-ind and $\operatorname{ind} F \leq 0$.

Then T is finite-ind and $\operatorname{ind} T \leq 0$.

In the sequel T_4 is a metrizable topological space.

We now state four propositions:

(37) Let A, B be closed subsets of T_4. Suppose A misses B. Let N_1 be a finite-ind subset of T_4. Suppose $\operatorname{ind} N_1 \leq 0$ and $T_4 \restriction N_1$ is second-countable. Then there exists a subset L of T_4 such that L separates A, B and L misses N_1.

(38) Let N_1 be a subset of T_4. Suppose $T_4 \restriction N_1$ is second-countable. Then N_1 is finite-ind and $\operatorname{ind} N_1 \leq 0$ if and only if for every point p of T_4 and for every open subset U of T_4 such that $p \in U$ there exists an open subset W of T_4 such that $p \in W$ and $W \subseteq U$ and N_1 misses $\operatorname{Fr} W$.

(39) Let N_1 be a subset of T_4. Suppose $T_4 \restriction N_1$ is second-countable. Then N_1 is finite-ind and $\operatorname{ind} N_1 \leq 0$ if and only if there exists a basis B of T_4 such that for every subset A of T_4 such that $A \in B$ holds N_1 misses $\operatorname{Fr} A$.

(40) Let N_1, A be subsets of T_4. Suppose $T_4 \restriction N_1$ is second-countable and N_1 is finite-ind and A is finite-ind and $\operatorname{ind} N_1 \leq 0$. Then $A \cup N_1$ is finite-ind and $\operatorname{ind}(A \cup N_1) \leq \operatorname{ind} A + 1$.

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