Basic Properties of Metrizable Topological Spaces

Karol Pąk
Institute of Computer Science
University of Białystok
Poland

Summary. We continue Mizar formalization of general topology according to the book [11] by Engelking. In the article, we present the final theorem of Section 4.1. Namely, the paper includes the formalization of theorems on the correspondence between the cardinalities of the basis and of some open subcover, and a discreet (closed) subspaces, and the weight of that metrizable topological space. We also define Lindelöf spaces and state the above theorem in this special case. We also introduce the concept of separation among two subsets (see [12]).

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The articles [21], [13], [20], [2], [1], [3], [10], [9], [7], [16], [4], [6], [19], [23], [22], [17], [15], [14], [8], [18], and [5] provide the notation and terminology for this paper.

1. Preliminaries

For simplicity, we follow the rules: $T$, $T_1$, $T_2$ denote topological spaces, $A$, $B$ denote subsets of $T$, $F$, $G$ denote families of subsets of $T$, $A_1$ denotes a subset of $T_1$, $A_2$ denotes a subset of $T_2$, $T_3$, $T_4$, $T_5$ denote metrizable topological spaces, $A_3$, $B_1$ denote subsets of $T_3$, $F_1$, $G_1$ denote families of subsets of $T_3$, $C$ denotes a cardinal number, and $i_1$ denotes an infinite cardinal number.

Let us consider $T_1$, $T_2$, $A_1$, $A_2$. We say that $A_1$ and $A_2$ are homeomorphic if and only if:

(Def. 1) $T_1 | A_1$ and $T_2 | A_2$ are homeomorphic.
Next we state four propositions:

1. $T_1$ and $T_2$ are homeomorphic iff $\Omega(T_1)$ and $\Omega(T_2)$ are homeomorphic.

2. Let $f$ be a function from $T_1$ into $T_2$. Suppose $f$ is homeomorphism.
   Let $g$ be a function from $T_1|A_1$ into $T_2|f^o A_1$. If $g = f|A_1$, then $g$ is homeomorphism.

3. For every function $f$ from $T_1$ into $T_2$ such that $f$ is homeomorphism holds $A_1$ and $f^o A_1$ are homeomorphic.

4. If $T_1$ and $T_2$ are homeomorphic, then weight $T_1 = weight T_2$.

Note that every topological space which is empty is also metrizable and every topological space which is metrizable is also $T_4$ and non empty. Let $M$ be a metric space. Note that $M_{top}$ is metrizable.

Let us consider $T_3$, $A_3$. Observe that $T_3|A_3$ is metrizable.

Next we state two propositions:

5. weight $T_1 \times T_2 \subseteq$ weight $T_1 \cdot$ weight $T_2$.

6. If $T_1$ is non empty and $T_2$ is non empty, then weight $T_1 \subseteq$ weight $T_1 \times T_2$ and weight $T_2 \subseteq$ weight $T_1 \times T_2$.

Let $T_1$, $T_2$ be second-countable topological spaces. One can check that $T_1 \times T_2$ is second-countable.

One can prove the following propositions:

7. Card($F|A)$) $\subseteq$ Card $F$.

8. For every basis $B_1$ of $T$ holds $B_1|A$ is a basis of $T|A$.

   Let $T$ be a second-countable topological space and let $A$ be a subset of $T$.

Note that $T|A$ is second-countable.

Let $M$ be a non empty metric space and let $A$ be a non empty subset of $M_{top}$. One can check that dist$_{min}(A)$ is continuous.

We now state the proposition

9. For every subset $B$ of $T$ and for every subset $F$ of $T|A$ such that $F = B$ holds $T|A|F = T|B$.

   Let us consider $T_3$. Observe that every subset of $T_3$ which is open is also $F_\sigma$ and every subset of $T_3$ which is closed is also $G_\delta$.

The following propositions are true:

10. For every subset $F$ of $T|B$ such that $A$ is $F_\sigma$ and $F = A \cap B$ holds $F$ is $F_\sigma$.

11. For every subset $F$ of $T|B$ such that $A$ is $G_\delta$ and $F = A \cap B$ holds $F$ is $G_\delta$.

12. If $T$ is a $T_1$ space and $A$ is discrete, then $A$ is an open subset of $T|\bar{A}$.

13. Let given $T$. Suppose that for every $F$ such that $F$ is open and a cover of $T$ there exists $G$ such that $G \subseteq F$ and $G$ is a cover of $T$ and Card $G \subseteq C$.
Let given $A$. If $A$ is closed and discrete, then $\text{Card } A \subseteq C$.

(14) Let given $T_3$. Suppose that for every $A_3$ such that $A_3$ is closed and discrete holds $\text{Card } A_3 \subseteq i_1$. Let given $A_3$. If $A_3$ is discrete, then $\text{Card } A_3 \subseteq i_1$.

(15) Let given $T$. Suppose that for every $A$ such that $A$ is discrete holds $\text{Card } A \subseteq C$. Let given $F$. Suppose $F$ is open and $\emptyset \notin F$ and for all $A, B$ such that $A, B \in F$ and $A \neq B$ holds $A$ misses $B$. Then $\text{Card } F \subseteq C$.

(16) For every $F$ such that $F$ is a cover of $T$ there exists $G$ such that $G \subseteq F$ and $G$ is a cover of $T$ and $\text{Card } G \subseteq \text{Card}(\Omega_T)$.

(17) If $A_3$ is dense, then weight $T_3 \subseteq \text{Card } \omega \cdot \text{Card } A_3$.

2. Main Properties

Next we state several propositions:

(18) weight $T_3 \subseteq i_1$ if and only if for every $F_1$ such that $F_1$ is open and a cover of $T_3$ there exists $G_1$ such that $G_1 \subseteq F_1$ and $G_1$ is a cover of $T_3$ and $\text{Card } G_1 \subseteq i_1$.

(19) weight $T_3 \subseteq i_1$ iff for every $A_3$ such that $A_3$ is closed and discrete holds $\text{Card } A_3 \subseteq i_1$.

(20) weight $T_3 \subseteq i_1$ iff for every $A_3$ such that $A_3$ is discrete holds $\text{Card } A_3 \subseteq i_1$.

(21) weight $T_3 \subseteq i_1$ if and only if for every $F_1$ such that $F_1$ is open and $\emptyset \notin F_1$ and for all $A_3, B_1$ such that $A_3, B_1 \in F_1$ and $A_3 \neq B_1$ holds $A_3$ misses $B_1$ holds $\text{Card } F_1 \subseteq i_1$.

(22) weight $T_3 \subseteq i_1$ iff density $T_3 \subseteq i_1$.

(23) Let $B$ be a basis of $T_3$. Suppose that for every $F_1$ such that $F_1$ is open and a cover of $T_3$ there exists $G_1$ such that $G_1 \subseteq F_1$ and $G_1$ is a cover of $T_3$ and $\text{Card } G_1 \subseteq i_1$. Then there exists a basis $u_1$ of $T_3$ such that $u_1 \subseteq B$ and $\text{Card } u_1 \subseteq i_1$.

3. Properties of Lindelöf spaces

Let us consider $T$. We say that $T$ is Lindelöf if and only if:

(Def. 2) For every $F$ such that $F$ is open and a cover of $T$ there exists $G$ such that $G \subseteq F$ and $G$ is a cover of $T$ and countable.

Next we state the proposition

(24) For every basis $B$ of $T_3$ such that $T_3$ is Lindelöf there exists a basis $B'$ of $T_3$ such that $B' \subseteq B$ and $B'$ is countable.

Let us observe that every metrizable topological space which is Lindelöf is also second-countable.
Let us note that every metrizable topological space which is Lindelöf is also separable and every metrizable topological space which is separable is also Lindelöf.

One can verify the following observations:

* there exists a non empty topological space which is Lindelöf and metrizable,
* every topological space which is second-countable is also Lindelöf,
* every topological space which is $T_3$ and Lindelöf is also $T_4$, and
* every topological space which is countable is also Lindelöf.

Let $n$ be a natural number. Note that the topological structure of $E^n$ is second-countable.

Let $T$ be a Lindelöf topological space and let $A$ be a closed subset of $T$. One can verify that $T \mid A$ is Lindelöf.

Let $T_3$ be a Lindelöf metrizable topological space and let $A$ be a subset of $T_3$. One can verify that $T_3 \mid A$ is Lindelöf.

Let us consider $T$ and let $A$, $B$, $L$ be subsets of $T$. We say that $L$ separates $A$, $B$ if and only if:

(Def. 3) There exist open subsets $U$, $W$ of $T$ such that $A \subseteq U$ and $B \subseteq W$ and $U$ misses $W$ and $L = (U \cup W)^c$.

The following two propositions are true:

(25) If $A_3$ and $B_1$ are separated, then there exists a subset $L$ of $T_3$ such that $L$ separates $A_3$, $B_1$.

(26) Let $M$ be a subset of $T_3$, $A_1$, $A_2$ be closed subsets of $T_3$, and $V_1$, $V_2$ be open subsets of $T_3$. Suppose $A_1 \subseteq V_1$ and $A_2 \subseteq V_2$ and $V_1$ misses $V_2$. Let $m_1$, $m_2$, $m_3$ be subsets of $T_3 \mid M$. Suppose $m_1 = M \cap V_1$ and $m_2 = M \cap V_2$ and $m_3$ separates $m_1$, $m_2$. Then there exists a subset $L$ of $T_3$ such that $L$ separates $A_1$, $A_2$ and $M \cap L \subseteq m_3$.

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