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Basic Properties of Metrizable Topological Spaces

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Summary. We continue Mizar formalization of general topology according to the book [11] by Engelking. In the article, we present the final theorem of Section 4.1. Namely, the paper includes the formalization of theorems on the correspondence between the cardinalities of the basis and of some open subcover, and a discreet (closed) subspaces, and the weight of that metrizable topological space. We also define Lindelöf spaces and state the above theorem in this special case. We also introduce the concept of separation among two subsets (see [12]).

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The articles [21], [13], [20], [2], [1], [3], [10], [9], [7], [16], [4], [6], [19], [23], [22], [17], [15], [14], [8], [18], and [5] provide the notation and terminology for this paper.

1. Preliminaries

For simplicity, we follow the rules: T, T_1, T_2 denote topological spaces, A, B denote subsets of T, F, G denote families of subsets of T, A_1 denotes a subset of T_1, A_2 denotes a subset of T_2, T_3, T_4, T_5 denote metrizable topological spaces, A_3, B_1 denote subsets of T_3, F_1, G_1 denote families of subsets of T_3, C denotes a cardinal number, and i_1 denotes an infinite cardinal number.

Let us consider T_1, T_2, A_1, A_2. We say that A_1 and A_2 are homeomorphic if and only if:

(Def. 1) $T_1 \restriction A_1$ and $T_2 \restriction A_2$ are homeomorphic.

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Next we state four propositions:

(1) T_1 and T_2 are homeomorphic iff $\Omega_{(T_1)}$ and $\Omega_{(T_2)}$ are homeomorphic.

(2) Let f be a function from T_1 into T_2. Suppose f is homeomorphism. Let g be a function from $T_1 \restriction A_1$ into $T_2 \restriction f^{\circ} A_1$. If $g = f \restriction A_1$, then g is homeomorphism.

(3) For every function f from T_1 into T_2 such that f is homeomorphism holds A_1 and $f^{\circ} A_1$ are homeomorphic.

(4) If T_1 and T_2 are homeomorphic, then weight $T_1 =$ weight T_2.

Note that every topological space which is empty is also metrizable and every topological space which is metrizable is also T_4 and non empty. Let M be a metric space. Note that M_{top} is metrizable.

Let us consider T_3, A_3. Observe that $T_3 \restriction A_3$ is metrizable.

Let us consider T_4, T_5. Observe that $T_4 \times T_5$ is metrizable.

Next we state two propositions:

(5) weight $T_1 \times T_2 \subseteq$ weight $T_1 \cdot$ weight T_2.

(6) If T_1 is non empty and T_2 is non empty, then weight $T_1 \subseteq$ weight $T_1 \times T_2$ and weight $T_2 \subseteq$ weight $T_1 \times T_2$.

Let T_1, T_2 be second-countable topological spaces. One can check that $T_1 \times T_2$ is second-countable.

One can prove the following propositions:

(7) $\mathrm{Card}(F \restriction A) \subseteq \mathrm{Card}\, F$.

(8) For every basis B_2 of T holds $B_2 \restriction A$ is a basis of $T \restriction A$.

Let T be a second-countable topological space and let A be a subset of T. Note that $T \restriction A$ is second-countable.

Let M be a non empty metric space and let A be a non empty subset of M_{top}. One can check that $\mathrm{dist}_{\min}(A)$ is continuous.

We now state the proposition

(9) For every subset B of T and for every subset F of $T \restriction A$ such that $F = B$ holds $T \restriction A \restriction F = T \restriction B$.

Let us consider T_3. Observe that every subset of T_3 which is open is also F_σ and every subset of T_3 which is closed is also G_δ.

The following propositions are true:

(10) For every subset F of $T \restriction B$ such that A is F_σ and $F = A \cap B$ holds F is F_σ.

(11) For every subset F of $T \restriction B$ such that A is G_δ and $F = A \cap B$ holds F is G_δ.

(12) If T is a T_1 space and A is discrete, then A is an open subset of $T \restriction \overline{A}$.

(13) Let given T. Suppose that for every F such that F is open and a cover of T there exists G such that $G \subseteq F$ and G is a cover of T and $\mathrm{Card}\, G \subseteq C$.

Let given A. If A is closed and discrete, then $\mathrm{Card}\, A \subseteq C$.

(14) Let given T_3. Suppose that for every A_3 such that A_3 is closed and discrete holds $\mathrm{Card}\, A_3 \subseteq i_1$. Let given A_3. If A_3 is discrete, then $\mathrm{Card}\, A_3 \subseteq i_1$.

(15) Let given T. Suppose that for every A such that A is discrete holds $\mathrm{Card}\, A \subseteq C$. Let given F. Suppose F is open and $\emptyset \notin F$ and for all A, B such that A, $B \in F$ and $A \neq B$ holds A misses B. Then $\mathrm{Card}\, F \subseteq C$.

(16) For every F such that F is a cover of T there exists G such that $G \subseteq F$ and G is a cover of T and $\mathrm{Card}\, G \subseteq \mathrm{Card}(\Omega_T)$.

(17) If A_3 is dense, then $\mathrm{weight}\, T_3 \subseteq \mathrm{Card}\, \omega \cdot \mathrm{Card}\, A_3$.

2. Main Properties

Next we state several propositions:

(18) $\mathrm{weight}\, T_3 \subseteq i_1$ if and only if for every F_1 such that F_1 is open and a cover of T_3 there exists G_1 such that $G_1 \subseteq F_1$ and G_1 is a cover of T_3 and $\mathrm{Card}\, G_1 \subseteq i_1$.

(19) $\mathrm{weight}\, T_3 \subseteq i_1$ iff for every A_3 such that A_3 is closed and discrete holds $\mathrm{Card}\, A_3 \subseteq i_1$.

(20) $\mathrm{weight}\, T_3 \subseteq i_1$ iff for every A_3 such that A_3 is discrete holds $\mathrm{Card}\, A_3 \subseteq i_1$.

(21) $\mathrm{weight}\, T_3 \subseteq i_1$ if and only if for every F_1 such that F_1 is open and $\emptyset \notin F_1$ and for all A_3, B_1 such that A_3, $B_1 \in F_1$ and $A_3 \neq B_1$ holds A_3 misses B_1 holds $\mathrm{Card}\, F_1 \subseteq i_1$.

(22) $\mathrm{weight}\, T_3 \subseteq i_1$ iff $\mathrm{density}\, T_3 \subseteq i_1$.

(23) Let B be a basis of T_3. Suppose that for every F_1 such that F_1 is open and a cover of T_3 there exists G_1 such that $G_1 \subseteq F_1$ and G_1 is a cover of T_3 and $\mathrm{Card}\, G_1 \subseteq i_1$. Then there exists a basis u_1 of T_3 such that $u_1 \subseteq B$ and $\mathrm{Card}\, u_1 \subseteq i_1$.

3. Properties of Lindelöf spaces

Let us consider T. We say that T is Lindelöf if and only if:

(Def. 2) For every F such that F is open and a cover of T there exists G such that $G \subseteq F$ and G is a cover of T and countable.

Next we state the proposition

(24) For every basis B of T_3 such that T_3 is Lindelöf there exists a basis B' of T_3 such that $B' \subseteq B$ and B' is countable.

Let us observe that every metrizable topological space which is Lindelöf is also second-countable.

Let us note that every metrizable topological space which is Lindelöf is also separable and every metrizable topological space which is separable is also Lindelöf.

One can verify the following observations:

* there exists a non empty topological space which is Lindelöf and metrizable,
* every topological space which is second-countable is also Lindelöf,
* every topological space which is T_3 and Lindelöf is also T_4, and
* every topological space which is countable is also Lindelöf.

Let n be a natural number. Note that the topological structure of $\mathcal{E}^n_{\rm T}$ is second-countable.

Let T be a Lindelöf topological space and let A be a closed subset of T. One can verify that $T{\restriction}A$ is Lindelöf.

Let T_3 be a Lindelöf metrizable topological space and let A be a subset of T_3. One can verify that $T_3{\restriction}A$ is Lindelöf.

Let us consider T and let A, B, L be subsets of T. We say that L separates A, B if and only if:

(Def. 3) There exist open subsets U, W of T such that $A \subseteq U$ and $B \subseteq W$ and U misses W and $L = (U \cup W)^{\rm c}$.

The following two propositions are true:

(25) If A_3 and B_1 are separated, then there exists a subset L of T_3 such that L separates A_3, B_1.

(26) Let M be a subset of T_3, A_1, A_2 be closed subsets of T_3, and V_1, V_2 be open subsets of T_3. Suppose $A_1 \subseteq V_1$ and $A_2 \subseteq V_2$ and $\overline{V_1}$ misses $\overline{V_2}$. Let m_1, m_2, m_3 be subsets of $T_3{\restriction}M$. Suppose $m_1 = M \cap \overline{V_1}$ and $m_2 = M \cap \overline{V_2}$ and m_3 separates m_1, m_2. Then there exists a subset L of T_3 such that L separates A_1, A_2 and $M \cap L \subseteq m_3$.

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