Basic Properties of Even and Odd Functions

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Summary. In this article we present definitions, basic properties and some examples of even and odd functions [6].

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The articles [2], [5], [1], [8], [14], [12], [15], [7], [17], [3], [4], [11], [19], [13], [10], [18], [16], and [9] provide the notation and terminology for this paper.

1. Even and Odd Functions

In this paper $x, r$ denote real numbers.
Let $A$ be a set. We say that $A$ is symmetrical if and only if:
(Def. 1) For every complex number $x$ such that $x \in A$ holds $-x \in A$.
One can check that there exists a subset of $\mathbb{C}$ which is symmetrical.
Let us note that there exists a subset of $\mathbb{R}$ which is symmetrical.
In the sequel $A$ is a symmetrical subset of $\mathbb{C}$.
Let $R$ be a binary relation. We say that $R$ has symmetrical domain if and only if:
(Def. 2) $\text{dom} \ R$ is symmetrical.
Let us observe that every binary relation which is empty has also symmetrical domain and there exists a binary relation which has symmetrical domain.
Let $R$ be a binary relation with symmetrical domain. One can check that $\text{dom} \ R$ is symmetrical.
Let $X, Y$ be complex-membered sets and let $F$ be a partial function from $X$ to $Y$. We say that $F$ is quasi even if and only if:
(Def. 3) For every $x$ such that $x, -x \in \text{dom } F$ holds $F(-x) = F(x)$.

Let $X, Y$ be complex-membered sets and let $F$ be a partial function from $X$ to $Y$. We say that $F$ is even if and only if:

(Def. 4) $F$ is quasi even and has symmetrical domain.

Let $X, Y$ be complex-membered sets. Note that every partial function from $X$ to $Y$ which is quasi even and has symmetrical domain is also even and every partial function from $X$ to $Y$ which is even is also quasi even and has symmetrical domain.

Let $A$ be a set, let $X, Y$ be complex-membered sets, and let $F$ be a partial function from $X$ to $Y$. We say that $F$ is even on $A$ if and only if:

(Def. 5) $A \subseteq \text{dom } F$ and $F|A$ is even.

Let $X, Y$ be complex-membered sets and let $F$ be a partial function from $X$ to $Y$. We say that $F$ is quasi odd if and only if:

(Def. 6) For every $x$ such that $x, -x \in \text{dom } F$ holds $F(-x) = -F(x)$.

Let $X, Y$ be complex-membered sets and let $F$ be a partial function from $X$ to $Y$. We say that $F$ is odd if and only if:

(Def. 7) $F$ is quasi odd and has symmetrical domain.

Let $X, Y$ be complex-membered sets. Note that every partial function from $X$ to $Y$ which is quasi odd and has symmetrical domain is also odd and every partial function from $X$ to $Y$ which is odd is also quasi odd and has symmetrical domain.

Let $A$ be a set, let $X, Y$ be complex-membered sets, and let $F$ be a partial function from $X$ to $Y$. We say that $F$ is odd on $A$ if and only if:

(Def. 8) $A \subseteq \text{dom } F$ and $F|A$ is odd.

In the sequel $F, G$ denote partial functions from $\mathbb{R}$ to $\mathbb{R}$.

One can prove the following propositions:

(1) $F$ is odd on $A$ iff $A \subseteq \text{dom } F$ and for every $x$ such that $x \in A$ holds $F(x) + F(-x) = 0$.

(2) $F$ is even on $A$ iff $A \subseteq \text{dom } F$ and for every $x$ such that $x \in A$ holds $F(x) - F(-x) = 0$.

(3) If $F$ is odd on $A$ and for every $x$ such that $x \in A$ holds $F(x) \neq 0$, then $A \subseteq \text{dom } F$ and for every $x$ such that $x \in A$ holds $\frac{F(x)}{F(-x)} = -1$.

(4) If $A \subseteq \text{dom } F$ and for every $x$ such that $x \in A$ holds $\frac{F(x)}{F(-x)} = -1$, then $F$ is odd on $A$.

(5) If $F$ is even on $A$ and for every $x$ such that $x \in A$ holds $F(x) \neq 0$, then $A \subseteq \text{dom } F$ and for every $x$ such that $x \in A$ holds $\frac{F(x)}{F(-x)} = 1$.

(6) If $A \subseteq \text{dom } F$ and for every $x$ such that $x \in A$ holds $\frac{F(x)}{F(-x)} = 1$, then $F$ is even on $A$. 
(7) If $F$ is even on $A$ and odd on $A$, then for every $x$ such that $x \in A$ holds $F(x) = 0$.
(8) If $F$ is even on $A$, then for every $x$ such that $x \in A$ holds $F(x) = F(|x|)$.
(9) If $A \subseteq \text{dom} F$ and for every $x$ such that $x \in A$ holds $F(x) = F(|x|)$, then $F$ is even on $A$.
(10) If $F$ is odd on $A$ and $G$ is odd on $A$, then $F + G$ is odd on $A$.
(11) If $F$ is even on $A$ and $G$ is even on $A$, then $F + G$ is even on $A$.
(12) If $F$ is odd on $A$ and $G$ is odd on $A$, then $F - G$ is odd on $A$.
(13) If $F$ is even on $A$ and $G$ is even on $A$, then $F - G$ is even on $A$.
(14) If $F$ is odd on $A$, then $rF$ is odd on $A$.
(15) If $F$ is even on $A$, then $rF$ is even on $A$.
(16) If $F$ is odd on $A$, then $-F$ is odd on $A$.
(17) If $F$ is even on $A$, then $-F$ is even on $A$.
(18) If $F$ is odd on $A$, then $F^{-1}$ is odd on $A$.
(19) If $F$ is even on $A$, then $F^{-1}$ is even on $A$.
(20) If $F$ is odd on $A$, then $|F|$ is even on $A$.
(21) If $F$ is even on $A$, then $|F|$ is even on $A$.
(22) If $F$ is odd on $A$ and $G$ is odd on $A$, then $FG$ is even on $A$.
(23) If $F$ is even on $A$ and $G$ is even on $A$, then $FG$ is even on $A$.
(24) If $F$ is odd on $A$ and $G$ is odd on $A$, then $FG$ is odd on $A$.
(25) If $F$ is even on $A$, then $r + F$ is even on $A$.
(26) If $F$ is even on $A$, then $F - r$ is even on $A$.
(27) If $F$ is even on $A$, then $F^2$ is even on $A$.
(28) If $F$ is odd on $A$, then $F^2$ is even on $A$.
(29) If $F$ is odd on $A$ and $G$ is odd on $A$, then $F/G$ is even on $A$.
(30) If $F$ is even on $A$ and $G$ is even on $A$, then $F/G$ is even on $A$.
(31) If $F$ is odd on $A$ and $G$ is even on $A$, then $F/G$ is odd on $A$.
(32) If $F$ is even on $A$ and $G$ is odd on $A$, then $F/G$ is odd on $A$.
(33) If $F$ is odd, then $-F$ is odd.
(34) If $F$ is even, then $-F$ is even.
(35) If $F$ is odd, then $F^{-1}$ is odd.
(36) If $F$ is even, then $F^{-1}$ is even.
(37) If $F$ is odd, then $|F|$ is even.
(38) If $F$ is even, then $|F|$ is even.
(39) If $F$ is odd, then $F^2$ is even.
(40) If $F$ is even, then $F^2$ is even.
(41) If $F$ is even, then $r + F$ is even.
(42) If $F$ is even, then $F - r$ is even.
(43) If $F$ is odd, then $r F$ is odd.
(44) If $F$ is even, then $r F$ is even.
(45) If $F$ is odd and $G$ is odd and $\text{dom} \ F \cap \text{dom} G$ is symmetrical, then $F + G$ is odd.
(46) If $F$ is even and $G$ is even and $\text{dom} \ F \cap \text{dom} G$ is symmetrical, then $F + G$ is even.
(47) If $F$ is odd and $G$ is odd and $\text{dom} \ F \cap \text{dom} G$ is symmetrical, then $F - G$ is odd.
(48) If $F$ is even and $G$ is even and $\text{dom} \ F \cap \text{dom} G$ is symmetrical, then $F - G$ is even.
(49) If $F$ is odd and $G$ is odd and $\text{dom} \ F \cap \text{dom} G$ is symmetrical, then $F G$ is even.
(50) If $F$ is even and $G$ is even and $\text{dom} \ F \cap \text{dom} G$ is symmetrical, then $F G$ is even.
(51) If $F$ is even and $G$ is odd and $\text{dom} \ F \cap \text{dom} G$ is symmetrical, then $F G$ is even.
(52) If $F$ is odd and $G$ is odd and $\text{dom} \ F \cap \text{dom} G$ is symmetrical, then $F G$ is odd.
(53) If $F$ is even and $G$ is even and $\text{dom} \ F \cap \text{dom} G$ is symmetrical, then $F G$ is odd.
(54) If $F$ is odd and $G$ is odd and $\text{dom} \ F \cap \text{dom} G$ is symmetrical, then $F G$ is odd.
(55) If $F$ is even and $G$ is odd and $\text{dom} \ F \cap \text{dom} G$ is symmetrical, then $F G$ is odd.

2. Some Examples

The function signum from $\mathbb{R}$ into $\mathbb{R}$ is defined by:

(Def. 9) For every real number $x$ holds $\text{signum}(x) = \text{sgn} x$.

Let $x$ be a real number. One can verify that $\text{signum}(x)$ is real.

Next we state a number of propositions:

(56) For every real number $x$ such that $x > 0$ holds $\text{signum}(x) = 1$.
(57) For every real number $x$ such that $x < 0$ holds $\text{signum}(x) = -1$.
(58) $\text{signum}(0) = 0$.
(59) For every real number $x$ holds $\text{signum}(x) = -\text{signum}(x)$.
(60) For every symmetrical subset $A$ of $\mathbb{R}$ holds $\text{signum}$ is odd on $A$.
(61) For every real number $x$ such that $x \geq 0$ holds $\|x\|_{\mathbb{R}}(x) = x$. 


(62) For every real number $x$ such that $x < 0$ holds $|\square|_\mathbb{R}(x) = -x$.

(63) For every real number $x$ holds $|\square|_\mathbb{R}(-x) = |\square|_\mathbb{R}(x)$.

(64) For every symmetrical subset $A$ of $\mathbb{R}$ holds $|\square|_\mathbb{R}$ is even on $A$.

(65) For every symmetrical subset $A$ of $\mathbb{R}$ holds the function $\sin$ is odd on $A$.

(66) For every symmetrical subset $A$ of $\mathbb{R}$ holds the function $\cos$ is even on $A$.

Let us observe that the function $\sin$ is odd.
Let us observe that the function $\cos$ is even.

We now state two propositions:

(67) For every symmetrical subset $A$ of $\mathbb{R}$ holds the function $\sinh$ is odd on $A$.

(68) For every symmetrical subset $A$ of $\mathbb{R}$ holds the function $\cosh$ is even on $A$.

Let us note that the function $\sinh$ is odd.
Let us mention that the function $\cosh$ is even.

The following propositions are true:

(69) If $A \subseteq [-\pi/2, \pi/2]$, then the function $\tan$ is odd on $A$.

(70) Suppose $A \subseteq \text{dom}(\text{the function tan})$ and for every $x$ such that $x \in A$ holds $(\text{the function cos})(x) \neq 0$. Then the function $\tan$ is odd on $A$.

(71) Suppose $A \subseteq \text{dom}(\text{the function cot})$ and for every $x$ such that $x \in A$ holds $(\text{the function sin})(x) \neq 0$. Then the function $\cot$ is odd on $A$.

(72) If $A \subseteq [-1, 1]$, then the function $\arctan$ is odd on $A$.

(73) For every symmetrical subset $A$ of $\mathbb{R}$ holds $|\text{the function sin}|$ is even on $A$.

(74) For every symmetrical subset $A$ of $\mathbb{R}$ holds $|\text{the function cos}|$ is even on $A$.

(75) For every symmetrical subset $A$ of $\mathbb{R}$ holds $(\text{the function sin})^{-1}$ is odd on $A$.

(76) For every symmetrical subset $A$ of $\mathbb{R}$ holds $(\text{the function cos})^{-1}$ is even on $A$.

(77) For every symmetrical subset $A$ of $\mathbb{R}$ holds $-\text{the function sin}$ is odd on $A$.

(78) For every symmetrical subset $A$ of $\mathbb{R}$ holds $-\text{the function cos}$ is even on $A$.

(79) For every symmetrical subset $A$ of $\mathbb{R}$ holds $(\text{the function sin})^2$ is even on $A$.

(80) For every symmetrical subset $A$ of $\mathbb{R}$ holds $(\text{the function cos})^2$ is even on $A$. 
In the sequel $B$ denotes a symmetrical subset of $\mathbb{R}$.

One can prove the following propositions:

(81) If $B \subseteq \text{dom}$ (the function sec), then the function sec is even on $B$.

(82) If for every real number $x$ such that $x \in B$ holds (the function cos)$(x) \neq 0$, then the function sec is even on $B$.

(83) If $B \subseteq \text{dom}$ (the function cosec), then the function cosec is odd on $B$.

(84) If for every real number $x$ such that $x \in B$ holds (the function sin)$(x) \neq 0$, then the function cosec is odd on $B$.

REFERENCES


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