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# Basic Properties of Even and Odd Functions

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**Summary.** In this article we present definitions, basic properties and some examples of even and odd functions [6].

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The articles [2], [5], [1], [8], [14], [12], [15], [7], [17], [3], [4], [11], [19], [13], [10], [18], [16], and [9] provide the notation and terminology for this paper.

## 1. Even and Odd Functions

In this paper $x$, $r$ denote real numbers.

Let $A$ be a set. We say that $A$ is symmetrical if and only if:

(Def. 1) For every complex number $x$ such that $x \in A$ holds $-x \in A$.

One can check that there exists a subset of $\mathbb{C}$ which is symmetrical.

Let us note that there exists a subset of $\mathbb{R}$ which is symmetrical.

In the sequel $A$ is a symmetrical subset of $\mathbb{C}$.

Let $R$ be a binary relation. We say that $R$ has symmetrical domain if and only if:

(Def. 2) $\operatorname{dom} R$ is symmetrical.

Let us observe that every binary relation which is empty has also symmetrical domain and there exists a binary relation which has symmetrical domain.

Let $R$ be a binary relation with symmetrical domain. One can check that $\operatorname{dom} R$ is symmetrical.

Let $X$, $Y$ be complex-membered sets and let $F$ be a partial function from $X$ to $Y$. We say that $F$ is quasi even if and only if:




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(Def. 3) For every $x$ such that $x$, $-x \in \operatorname{dom} F$ holds $F(-x) = F(x)$.

Let $X$, $Y$ be complex-membered sets and let $F$ be a partial function from $X$ to $Y$. We say that $F$ is even if and only if:

(Def. 4) $F$ is quasi even and has symmetrical domain.

Let $X$, $Y$ be complex-membered sets. Note that every partial function from $X$ to $Y$ which is quasi even and has symmetrical domain is also even and every partial function from $X$ to $Y$ which is even is also quasi even and has symmetrical domain.

Let $A$ be a set, let $X$, $Y$ be complex-membered sets, and let $F$ be a partial function from $X$ to $Y$. We say that $F$ is even on $A$ if and only if:

(Def. 5) $A \subseteq \operatorname{dom} F$ and $F{\restriction}A$ is even.

Let $X$, $Y$ be complex-membered sets and let $F$ be a partial function from $X$ to $Y$. We say that $F$ is quasi odd if and only if:

(Def. 6) For every $x$ such that $x$, $-x \in \operatorname{dom} F$ holds $F(-x) = -F(x)$.

Let $X$, $Y$ be complex-membered sets and let $F$ be a partial function from $X$ to $Y$. We say that $F$ is odd if and only if:

(Def. 7) $F$ is quasi odd and has symmetrical domain.

Let $X$, $Y$ be complex-membered sets. Note that every partial function from $X$ to $Y$ which is quasi odd and has symmetrical domain is also odd and every partial function from $X$ to $Y$ which is odd is also quasi odd and has symmetrical domain.

Let $A$ be a set, let $X$, $Y$ be complex-membered sets, and let $F$ be a partial function from $X$ to $Y$. We say that $F$ is odd on $A$ if and only if:

(Def. 8) $A \subseteq \operatorname{dom} F$ and $F{\restriction}A$ is odd.

In the sequel $F$, $G$ denote partial functions from $\mathbb{R}$ to $\mathbb{R}$.

One can prove the following propositions:

(1) $F$ is odd on $A$ iff $A \subseteq \operatorname{dom} F$ and for every $x$ such that $x \in A$ holds $F(x) + F(-x) = 0$.

(2) $F$ is even on $A$ iff $A \subseteq \operatorname{dom} F$ and for every $x$ such that $x \in A$ holds $F(x) - F(-x) = 0$.

(3) If $F$ is odd on $A$ and for every $x$ such that $x \in A$ holds $F(x) \neq 0$, then $A \subseteq \operatorname{dom} F$ and for every $x$ such that $x \in A$ holds $\frac{F(x)}{F(-x)} = -1$.

(4) If $A \subseteq \operatorname{dom} F$ and for every $x$ such that $x \in A$ holds $\frac{F(x)}{F(-x)} = -1$, then $F$ is odd on $A$.

(5) If $F$ is even on $A$ and for every $x$ such that $x \in A$ holds $F(x) \neq 0$, then $A \subseteq \operatorname{dom} F$ and for every $x$ such that $x \in A$ holds $\frac{F(x)}{F(-x)} = 1$.

(6) If $A \subseteq \operatorname{dom} F$ and for every $x$ such that $x \in A$ holds $\frac{F(x)}{F(-x)} = 1$, then $F$ is even on $A$.

(7) If $F$ is even on $A$ and odd on $A$, then for every $x$ such that $x \in A$ holds $F(x) = 0$.

(8) If $F$ is even on $A$, then for every $x$ such that $x \in A$ holds $F(x) = F(|x|)$.

(9) If $A \subseteq \operatorname{dom} F$ and for every $x$ such that $x \in A$ holds $F(x) = F(|x|)$, then $F$ is even on $A$.

(10) If $F$ is odd on $A$ and $G$ is odd on $A$, then $F + G$ is odd on $A$.

(11) If $F$ is even on $A$ and $G$ is even on $A$, then $F + G$ is even on $A$.

(12) If $F$ is odd on $A$ and $G$ is odd on $A$, then $F - G$ is odd on $A$.

(13) If $F$ is even on $A$ and $G$ is even on $A$, then $F - G$ is even on $A$.

(14) If $F$ is odd on $A$, then $r\,F$ is odd on $A$.

(15) If $F$ is even on $A$, then $r\,F$ is even on $A$.

(16) If $F$ is odd on $A$, then $-F$ is odd on $A$.

(17) If $F$ is even on $A$, then $-F$ is even on $A$.

(18) If $F$ is odd on $A$, then $F^{-1}$ is odd on $A$.

(19) If $F$ is even on $A$, then $F^{-1}$ is even on $A$.

(20) If $F$ is odd on $A$, then $|F|$ is even on $A$.

(21) If $F$ is even on $A$, then $|F|$ is even on $A$.

(22) If $F$ is odd on $A$ and $G$ is odd on $A$, then $F\,G$ is even on $A$.

(23) If $F$ is even on $A$ and $G$ is even on $A$, then $F\,G$ is even on $A$.

(24) If $F$ is even on $A$ and $G$ is odd on $A$, then $F\,G$ is odd on $A$.

(25) If $F$ is even on $A$, then $r + F$ is even on $A$.

(26) If $F$ is even on $A$, then $F - r$ is even on $A$.

(27) If $F$ is even on $A$, then $F^{\mathbf{2}}$ is even on $A$.

(28) If $F$ is odd on $A$, then $F^{\mathbf{2}}$ is even on $A$.

(29) If $F$ is odd on $A$ and $G$ is odd on $A$, then $F/G$ is even on $A$.

(30) If $F$ is even on $A$ and $G$ is even on $A$, then $F/G$ is even on $A$.

(31) If $F$ is odd on $A$ and $G$ is even on $A$, then $F/G$ is odd on $A$.

(32) If $F$ is even on $A$ and $G$ is odd on $A$, then $F/G$ is odd on $A$.

(33) If $F$ is odd, then $-F$ is odd.

(34) If $F$ is even, then $-F$ is even.

(35) If $F$ is odd, then $F^{-1}$ is odd.

(36) If $F$ is even, then $F^{-1}$ is even.

(37) If $F$ is odd, then $|F|$ is even.

(38) If $F$ is even, then $|F|$ is even.

(39) If $F$ is odd, then $F^{\mathbf{2}}$ is even.

(40) If $F$ is even, then $F^{\mathbf{2}}$ is even.

(41) If $F$ is even, then $r + F$ is even.

(42) If $F$ is even, then $F - r$ is even.

(43) If $F$ is odd, then $r\,F$ is odd.

(44) If $F$ is even, then $r\,F$ is even.

(45) If $F$ is odd and $G$ is odd and $\operatorname{dom} F \cap \operatorname{dom} G$ is symmetrical, then $F+G$ is odd.

(46) If $F$ is even and $G$ is even and $\operatorname{dom} F \cap \operatorname{dom} G$ is symmetrical, then $F+G$ is even.

(47) If $F$ is odd and $G$ is odd and $\operatorname{dom} F \cap \operatorname{dom} G$ is symmetrical, then $F-G$ is odd.

(48) If $F$ is even and $G$ is even and $\operatorname{dom} F \cap \operatorname{dom} G$ is symmetrical, then $F-G$ is even.

(49) If $F$ is odd and $G$ is odd and $\operatorname{dom} F \cap \operatorname{dom} G$ is symmetrical, then $F\,G$ is even.

(50) If $F$ is even and $G$ is even and $\operatorname{dom} F \cap \operatorname{dom} G$ is symmetrical, then $F\,G$ is even.

(51) If $F$ is even and $G$ is odd and $\operatorname{dom} F \cap \operatorname{dom} G$ is symmetrical, then $F\,G$ is odd.

(52) If $F$ is odd and $G$ is odd and $\operatorname{dom} F \cap \operatorname{dom} G$ is symmetrical, then $F/G$ is even.

(53) If $F$ is even and $G$ is even and $\operatorname{dom} F \cap \operatorname{dom} G$ is symmetrical, then $F/G$ is even.

(54) If $F$ is odd and $G$ is even and $\operatorname{dom} F \cap \operatorname{dom} G$ is symmetrical, then $F/G$ is odd.

(55) If $F$ is even and $G$ is odd and $\operatorname{dom} F \cap \operatorname{dom} G$ is symmetrical, then $F/G$ is odd.

## 2. Some Examples

The function signum from $\mathbb{R}$ into $\mathbb{R}$ is defined by:

(Def. 9) For every real number $x$ holds $\operatorname{signum}(x) = \operatorname{sgn} x$.

Let $x$ be a real number. One can verify that $\operatorname{signum}(x)$ is real.

Next we state a number of propositions:

(56) For every real number $x$ such that $x > 0$ holds $\operatorname{signum}(x) = 1$.

(57) For every real number $x$ such that $x < 0$ holds $\operatorname{signum}(x) = -1$.

(58) $\operatorname{signum}(0) = 0$.

(59) For every real number $x$ holds $\operatorname{signum}(-x) = -\operatorname{signum}(x)$.

(60) For every symmetrical subset $A$ of $\mathbb{R}$ holds signum is odd on $A$.

(61) For every real number $x$ such that $x \geq 0$ holds $|\square|_{\mathbb{R}}(x) = x$.

(62) For every real number $x$ such that $x < 0$ holds $|\Box|_{\mathbb{R}}(x) = -x$.

(63) For every real number $x$ holds $|\Box|_{\mathbb{R}}(-x) = |\Box|_{\mathbb{R}}(x)$.

(64) For every symmetrical subset $A$ of $\mathbb{R}$ holds $|\Box|_{\mathbb{R}}$ is even on $A$.

(65) For every symmetrical subset $A$ of $\mathbb{R}$ holds the function sin is odd on $A$.

(66) For every symmetrical subset $A$ of $\mathbb{R}$ holds the function cos is even on $A$.

Let us observe that the function sin is odd.

Let us observe that the function cos is even.

We now state two propositions:

(67) For every symmetrical subset $A$ of $\mathbb{R}$ holds the function sinh is odd on $A$.

(68) For every symmetrical subset $A$ of $\mathbb{R}$ holds the function cosh is even on $A$.

Let us note that the function sinh is odd.

Let us mention that the function cosh is even.

The following propositions are true:

(69) If $A \subseteq ]-\frac{\pi}{2}, \frac{\pi}{2}[$, then the function tan is odd on $A$.

(70) Suppose $A \subseteq \mathrm{dom}\,(\text{the function tan})$ and for every $x$ such that $x \in A$ holds (the function cos)$(x) \neq 0$. Then the function tan is odd on $A$.

(71) Suppose $A \subseteq \mathrm{dom}\,(\text{the function cot})$ and for every $x$ such that $x \in A$ holds (the function sin)$(x) \neq 0$. Then the function cot is odd on $A$.

(72) If $A \subseteq [-1, 1]$, then the function arctan is odd on $A$.

(73) For every symmetrical subset $A$ of $\mathbb{R}$ holds |the function sin | is even on $A$.

(74) For every symmetrical subset $A$ of $\mathbb{R}$ holds |the function cos | is even on $A$.

(75) For every symmetrical subset $A$ of $\mathbb{R}$ holds (the function sin) $^{-1}$ is odd on $A$.

(76) For every symmetrical subset $A$ of $\mathbb{R}$ holds (the function cos) $^{-1}$ is even on $A$.

(77) For every symmetrical subset $A$ of $\mathbb{R}$ holds $-$the function sin is odd on $A$.

(78) For every symmetrical subset $A$ of $\mathbb{R}$ holds $-$the function cos is even on $A$.

(79) For every symmetrical subset $A$ of $\mathbb{R}$ holds (the function sin)$^{\mathbf{2}}$ is even on $A$.

(80) For every symmetrical subset $A$ of $\mathbb{R}$ holds (the function cos)$^{\mathbf{2}}$ is even on $A$.

In the sequel $B$ denotes a symmetrical subset of $\mathbb{R}$.

One can prove the following propositions:

(81) If $B \subseteq \operatorname{dom}$ (the function sec), then the function sec is even on $B$.

(82) If for every real number $x$ such that $x \in B$ holds (the function cos)$(x) \neq 0$, then the function sec is even on $B$.

(83) If $B \subseteq \operatorname{dom}$ (the function cosec), then the function cosec is odd on $B$.

(84) If for every real number $x$ such that $x \in B$ holds (the function sin)$(x) \neq 0$, then the function cosec is odd on $B$.

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