Hopf Extension Theorem of Measure

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Summary. The authors have presented some articles about Lebesgue type integration theory. In our previous articles [12, 13, 26], we assumed that some \( \sigma \)-additive measure existed and that a function was measurable on that measure. However the existence of such a measure is not trivial. In general, because the construction of a finite additive measure is comparatively easy, to induce a \( \sigma \)-additive measure a finite additive measure is used. This is known as an E. Hopf’s extension theorem of measure [15].

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The articles [11], [23], [1], [24], [22], [8], [25], [10], [9], [2], [20], [26], [6], [5], [7], [13], [4], [12], [3], [16], [19], [18], [27], [21], [17], and [14] provide the notation and terminology for this paper.

1. The Outer Measure Induced by the Finite Additive Measure

For simplicity, we follow the rules: \( X \) denotes a set, \( F \) denotes a field of subsets of \( X \), \( M \) denotes a measure on \( F \), \( A, B \) denote subsets of \( X \), \( S_1 \) denotes a sequence of subsets of \( X \), \( s_1, s_2, s_3 \) denote sequences of extended reals, and \( n, k \) denote natural numbers.

We now state three propositions:

1. \[ \text{Ser} s_1 = (\sum_{\alpha=0}^{\kappa} (s_1)(\alpha))_{\kappa \in \mathbb{N}}. \]
2. If \( s_1 \) is non-negative, then \( s_1 \) is summable and \( \sum s_1 = \sum s_1. \)

\(^1\)The translation of Mizar functor SUM introduced in [4] was changed from \( \sum \) to \( \sum. \)
(3) Suppose $s_2$ is non-negative and $s_3$ is non-negative and for every natural number $n$ holds $s_1(n) = s_2(n) + s_3(n)$. Then $s_1$ is non-negative and $\sum s_1 = \sum s_2 + \sum s_3$.

Let us consider $X, F$. One can check that there exists a function from $\mathbb{N}$ into $F$ which is disjoint valued.

Let us consider $X, F$. A finite sequence of elements of $2^X$ is said to be a finite sequence of elements of $F$ if:

(Def. 1) For every natural number $k$ such that $k \in \text{dom } it$ holds $it(k) \in F$.

Let us consider $X, F$. Observe that there exists a finite sequence of elements of $F$ which is disjoint valued.

Let us consider $X, F$. A disjoint valued finite set sequence of $F$ is a disjoint valued finite sequence of elements of $F$.

Let us consider $X, F$. A sequence of separated subsets of $F$ is a disjoint valued function from $\mathbb{N}$ into $F$.

Let us consider $X, F$. A sequence of subsets of $X$ is said to be a set sequence of $F$ if:

(Def. 2) For every natural number $n$ holds $it(n) \in F$.

Let us consider $X, F, S_1$. A function from $\mathbb{N}$ into $(2^X)^\mathbb{N}$ is said to be a covering of $S_1$ in $F$ if:

(Def. 3) $A \subseteq \bigcup \text{rng } it$.

In the sequel $F_1$ denotes a set sequence of $F$ and $C_1$ denotes a covering of $A$ in $F$.

Let us consider $X, F, F_1, n$. Then $F_1(n)$ is an element of $F$.

Let us consider $X, F, S_1$. A function from $\mathbb{N}$ into $(2^X)^\mathbb{N}$ is said to be a covering of $S_1$ in $F$ if:

(Def. 4) For every element $n$ of $\mathbb{N}$ holds $it(n)$ is a covering of $S_1(n)$ in $F$.

In the sequel $C_2$ is a covering of $S_1$ in $F$.

Let us consider $X, F, M, F_1$. The functor $\text{vol}(M, F_1)$ yielding a sequence of extended reals is defined as follows:

(Def. 5) For every $n$ holds $(\text{vol}(M, F_1))(n) = M(F_1(n))$.

One can prove the following proposition

(4) $\text{vol}(M, F_1)$ is non-negative.

Let us consider $X, F, S_1, C_2$ and let $n$ be an element of $\mathbb{N}$. Then $C_2(n)$ is a covering of $S_1(n)$ in $F$.

Let us consider $X, F, S_1, M, C_2$. The functor $\text{Volume}(M, C_2)$ yielding a sequence of extended reals is defined as follows:

(Def. 6) For every element $n$ of $\mathbb{N}$ holds $(\text{Volume}(M, C_2))(n) = \sum \text{vol}(M, C_2(n))$.

The following proposition is true

(5) $0 \leq (\text{Volume}(M, C_2))(n)$. 
Let us consider $X$, $F$, $M$, $A$. The functor $\text{Svc}(M, A)$ yielding a subset of $\mathbb{R}$ is defined as follows:

(Def. 7) For every extended real number $x$ holds $x \in \text{Svc}(M, A)$ iff there exists a covering $C_1$ of $A$ in $F$ such that $x = \sum \text{vol}(M, C_1)$.

Let us consider $X$, $A$, $F$, $M$. Observe that $\text{Svc}(M, A)$ is non-empty.

Let us consider $X$, $F$, $M$. The Caratheodory measure determined by $M$ is a function from $2^X$ into $\mathbb{R}$ and is defined by:

(Def. 8) For every subset $A$ of $X$ holds (the Caratheodory measure determined by $M$)($A$) = $\inf \text{Svc}(M, A)$.

The function $\text{InvPairFunc}$ from $\mathbb{N}$ into $\mathbb{N} \times \mathbb{N}$ is defined by:

(Def. 9) $\text{InvPairFunc} = \text{PairFunc}^{-1}$.

Let us consider $X$, $F$, $S_1$, $C_2$. The functor $\text{On}C_2$ yielding a covering of $\bigcup \text{rng} S_1$ in $F$ is defined by:

(Def. 10) For every natural number $n$ holds ($\text{On}C_2$)(n) = $C_2(\text{pr1}(\text{InvPairFunc})(n))(\text{pr2}(\text{InvPairFunc})(n))$.

The following propositions are true:

(6) Let $k$ be an element of $\mathbb{N}$. Then there exists a natural number $m$ such that for every sequence $S_1$ of subsets of $X$ and for every covering $C_2$ of $S_1$ in $F$ holds ($\sum_{\kappa=0}^{\kappa}(\text{vol}(M, \text{On}C_2))(\alpha))_{\kappa \in \mathbb{N}}(k) \leq (\sum_{\alpha=0}^{\alpha}(\text{Volume}(M, C_2))(\alpha))_{\alpha \in \mathbb{N}}(m)$.

(7) $\inf \text{Svc}(M, \bigcup \text{rng} S_1) \leq \sum \text{Volume}(M, C_2)$.

(8) If $A \in F$, then $A, \emptyset_X$ followed by $\emptyset_X$ is a covering of $A$ in $F$.

(9) Let $X$ be a set, $F$ be a field of subsets of $X$, $M$ be a measure on $F$, and $A$ be a set. If $A \in F$, then (the Caratheodory measure determined by $M$)($A$) $\leq M(A)$.

(10) The Caratheodory measure determined by $M$ is non-negative.

(11) (The Caratheodory measure determined by $M$)($\emptyset$) = 0.

(12) If $A \subseteq B$, then (the Caratheodory measure determined by $M$)($A$) $\leq$ (the Caratheodory measure determined by $M$)($B$).

(13) (The Caratheodory measure determined by $M$)($\bigcup \text{rng} S_1$) $\leq \sum$((the Caratheodory measure determined by $M$)($S_1$)).

(14) The Caratheodory measure determined by $M$ is a Caratheodory’s measure on $X$.

Let $X$ be a set, let $F$ be a field of subsets of $X$, and let $M$ be a measure on $F$. Then the Caratheodory measure determined by $M$ is a Caratheodory’s measure on $X$. 
2. Hopf Extension Theorem

Let $X$ be a set, let $F$ be a field of subsets of $X$, and let $M$ be a measure on $F$. We say that $M$ is completely-additive if and only if:

(Def. 11) For every sequence $F_1$ of separated subsets of $F$ such that $\bigcup \operatorname{rng} F_1 \in F$ holds $\sum (M \cdot F_1) = M(\bigcup \operatorname{rng} F_1)$.

The following propositions are true:

(15) The partial unions of $F_1$ are a set sequence of $F$.
(16) The partial diff-unions of $F_1$ are a set sequence of $F$.
(17) Suppose $A \in F$. Then there exists a sequence $F_1$ of separated subsets of $F$ such that $A = \bigcup \operatorname{rng} F_1$ and for every natural number $n$ holds $F_1(n) \subseteq C_1(n)$.
(18) Suppose $M$ is completely-additive. Let $A$ be a set. If $A \in F$, then $M(A) = (\text{the Caratheodory measure determined by } M)(A)$.

In the sequel $C$ is a Carathéodory's measure on $X$.

We now state three propositions:

(19) If for every subset $B$ of $X$ holds $C(B \cap A) + C(B \cap (X \setminus A)) \leq C(B)$, then $A \in \sigma\text{-Field}(C)$.
(20) $F \subseteq \sigma\text{-Field(}\text{the Caratheodory measure determined by } M\text{)}$.
(21) Let $X$ be a set, $F$ be a field of subsets of $X$, $F_1$ be a set sequence of $F$, and $M$ be a function from $F$ into $\mathbb{R}$. Then $M \cdot F_1$ is a sequence of extended reals.

Let $X$ be a set, let $F$ be a field of subsets of $X$, let $F_1$ be a set sequence of $F$, and let $g$ be a function from $F$ into $\mathbb{R}$. Then $g \cdot F_1$ is a sequence of extended reals.

One can prove the following proposition

(22) Let $X$ be a set, $S$ be a $\sigma$-field of subsets of $X$, $S_2$ be a sequence of subsets of $S$, and $M$ be a function from $S$ into $\mathbb{R}$. Then $M \cdot S_2$ is a sequence of extended reals.

Let $X$ be a set, let $S$ be a $\sigma$-field of subsets of $X$, let $S_2$ be a sequence of subsets of $S$, and let $g$ be a function from $S$ into $\mathbb{R}$. Then $g \cdot S_2$ is a sequence of extended reals.

Next we state several propositions:

(23) Let $F, G$ be functions from $\mathbb{N}$ into $\mathbb{R}$ and $n$ be a natural number. Suppose that for every natural number $m$ such that $m \leq n$ holds $F(m) \leq G(m)$. Then $(\text{Ser } F)(n) \leq (\text{Ser } G)(n)$.
(24) For all $X, C$ and for every sequence $s_1$ of separated subsets of $\sigma\text{-Field}(C)$ holds $\bigcup \operatorname{rng} s_1 \in \sigma\text{-Field}(C)$ and $C(\bigcup \operatorname{rng} s_1) = \sum (C \cdot s_1)$.
(25) For all $X, C$ and for every sequence $s_1$ of subsets of $\sigma\text{-Field}(C)$ holds $\bigcup s_1 \in \sigma\text{-Field}(C)$.
(26) Let $X$ be a non empty set, $S$ be a σ-field of subsets of $X$, $M$ be a σ-measure on $S$, and $S_2$ be a sequence of subsets of $S$. If $S_2$ is non-decreasing, then $\lim (M \cdot S_2) = M(\lim S_2)$.

(27) If $F_1$ is non-decreasing, then $M \cdot F_1$ is non-decreasing.

(28) If $F_1$ is descending, then $M \cdot F_1$ is non-increasing.

(29) Let $X$ be a set, $S$ be a σ-field of subsets of $X$, $M$ be a σ-measure on $S$, and $S_2$ be a sequence of subsets of $S$. If $S_2$ is non-decreasing, then $M \cdot S_2$ is non-decreasing.

(30) Let $X$ be a set, $S$ be a σ-field of subsets of $X$, $M$ be a σ-measure on $S$, and $S_2$ be a sequence of subsets of $S$. If $S_2$ is descending, then $M \cdot S_2$ is non-increasing.

(31) Let $X$ be a non empty set, $S$ be a σ-field of subsets of $X$, $M$ be a σ-measure on $S$, and $S_2$ be a sequence of subsets of $S$. If $S_2$ is descending and $M(S_2(0)) < +\infty$, then $\lim (M \cdot S_2) = M(\lim S_2)$.

Let $X$ be a set, let $F$ be a field of subsets of $X$, let $S$ be a σ-field of subsets of $X$, let $m$ be a measure on $F$, and let $M$ be a σ-measure on $S$. We say that $M$ is an extension of $m$ if and only if:

(Def. 12) For every set $A$ such that $A \in F$ holds $M(A) = m(A)$.

The following four propositions are true:

(32) Let $X$ be a non empty set, $F$ be a field of subsets of $X$, and $m$ be a measure on $F$. If there exists a σ-measure on $\sigma(F)$ which is an extension of $m$, then $m$ is completely-additive.

(33) Let $X$ be a non empty set, $F$ be a field of subsets of $X$, and $m$ be a measure on $F$. Suppose $m$ is completely-additive. Then there exists a σ-measure $M$ on $\sigma(F)$ such that $M$ is an extension of $m$ and $M = \sigma\text{-Meas}(the\ Caratheodory\ measure\ determined\ by\ m)\mid\sigma(F)$.

(34) If for every $n$ holds $M(F_1(n)) < +\infty$, then $M((the\ partial\ unions\ of\ F_1)(k)) < +\infty$.

(35) Let $X$ be a non empty set, $F$ be a field of subsets of $X$, and $m$ be a measure on $F$. Suppose that

(i) $m$ is completely-additive, and

(ii) there exists a set sequence $A_1$ of $F$ such that for every natural number $n$ holds $m(A_1(n)) < +\infty$ and $X = \bigcup \text{rng} A_1$.

Let $M$ be a σ-measure on $\sigma(F)$. Suppose $M$ is an extension of $m$. Then $M = \sigma\text{-Meas}(the\ Caratheodory\ measure\ determined\ by\ m)\mid\sigma(F)$.

REFERENCES


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