FORMALIZED MATHEMATICS
Vol. 17, No. 2, Pages 157–162, 2009
DOI: 10.2478/v10037-009-0018-6

Hopf Extension Theorem of Measure

Noboru Endou
Gifu National College of Technology
Japan

Hiroyuki Okazaki
Shinshu University
Nagano, Japan

Yasunari Shidama
Shinshu University
Nagano, Japan

Summary. The authors have presented some articles about Lebesgue type integration theory. In our previous articles [12, 13, 26], we assumed that some σ-additive measure existed and that a function was measurable on that measure. However the existence of such a measure is not trivial. In general, because the construction of a finite additive measure is comparatively easy, to induce a σ-additive measure a finite additive measure is used. This is known as an E. Hopf's extension theorem of measure [15].

MML identifier: `MEASURE8`, version: `7.11.02 4.125.1059`

The articles [11], [23], [1], [24], [22], [8], [25], [10], [9], [2], [20], [26], [6], [5], [7], [13], [4], [12], [3], [16], [19], [18], [27], [21], [17], and [14] provide the notation and terminology for this paper.

1. The Outer Measure Induced by the Finite Additive Measure

For simplicity, we follow the rules: X denotes a set, F denotes a field of subsets of X, M denotes a measure on F, A, B denote subsets of X, S_1 denotes a sequence of subsets of X, s_1, s_2, s_3 denote sequences of extended reals, and n, k denote natural numbers.

We now state three propositions:

(1) $\operatorname{Ser} s_1 = (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$.

(2)[1] If s_1 is non-negative, then s_1 is summable and $\overline{\sum} s_1 = \sum s_1$.

[1]The translation of Mizar functor SUM introduced in [4] was changed from $\sum$ to $\overline{\sum}$.

ISSN 1426–2630(p), 1898-9934(e)

(3) Suppose s_2 is non-negative and s_3 is non-negative and for every natural number n holds $s_1(n) = s_2(n)+s_3(n)$. Then s_1 is non-negative and $\overline{\sum} s_1 = \overline{\sum} s_2 + \overline{\sum} s_3$ and $\sum s_1 = \sum s_2 + \sum s_3$.

Let us consider X, F. One can check that there exists a function from $\mathbb{N}$ into F which is disjoint valued.

Let us consider X, F. A finite sequence of elements of 2^X is said to be a finite sequence of elements of F if:

(Def. 1) For every natural number k such that $k \in \mathrm{dom}\,\mathrm{it}$ holds $\mathrm{it}(k) \in F$.

Let us consider X, F. Observe that there exists a finite sequence of elements of F which is disjoint valued.

Let us consider X, F. A disjoint valued finite set sequence of F is a disjoint valued finite sequence of elements of F.

Let us consider X, F. A sequence of separated subsets of F is a disjoint valued function from $\mathbb{N}$ into F.

Let us consider X, F. A sequence of subsets of X is said to be a set sequence of F if:

(Def. 2) For every natural number n holds $\mathrm{it}(n) \in F$.

Let us consider X, A, F. A set sequence of F is said to be a covering of A in F if:

(Def. 3) $A \subseteq \bigcup \mathrm{rng}\,\mathrm{it}$.

In the sequel F_1 denotes a set sequence of F and C_1 denotes a covering of A in F.

Let us consider X, F, F_1, n. Then $F_1(n)$ is an element of F.

Let us consider X, F, S_1. A function from $\mathbb{N}$ into $(2^X)^{\mathbb{N}}$ is said to be a covering of S_1 in F if:

(Def. 4) For every element n of $\mathbb{N}$ holds $\mathrm{it}(n)$ is a covering of $S_1(n)$ in F.

In the sequel C_2 is a covering of S_1 in F.

Let us consider X, F, M, F_1. The functor $\mathrm{vol}(M, F_1)$ yielding a sequence of extended reals is defined as follows:

(Def. 5) For every n holds $(\mathrm{vol}(M, F_1))(n) = M(F_1(n))$.

One can prove the following proposition

(4) $\mathrm{vol}(M, F_1)$ is non-negative.

Let us consider X, F, S_1, C_2 and let n be an element of $\mathbb{N}$. Then $C_2(n)$ is a covering of $S_1(n)$ in F.

Let us consider X, F, S_1, M, C_2. The functor $\mathrm{Volume}(M, C_2)$ yielding a sequence of extended reals is defined as follows:

(Def. 6) For every element n of $\mathbb{N}$ holds $(\mathrm{Volume}(M, C_2))(n) = \overline{\sum} \mathrm{vol}(M, C_2(n))$.

The following proposition is true

(5) $0 \leq (\mathrm{Volume}(M, C_2))(n)$.

Let us consider X, F, M, A. The functor $\mathrm{Svc}(M, A)$ yielding a subset of $\overline{\mathbb{R}}$ is defined as follows:

(Def. 7) For every extended real number x holds $x \in \mathrm{Svc}(M, A)$ iff there exists a covering C_1 of A in F such that $x = \overline{\sum} \mathrm{vol}(M, C_1)$.

Let us consider X, A, F, M. Observe that $\mathrm{Svc}(M, A)$ is non empty.

Let us consider X, F, M. The Caratheodory measure determined by M is a function from 2^X into $\overline{\mathbb{R}}$ and is defined by:

(Def. 8) For every subset A of X holds (the Caratheodory measure determined by $M)(A) = \inf \mathrm{Svc}(M, A)$.

The function InvPairFunc from $\mathbb{N}$ into $\mathbb{N} \times \mathbb{N}$ is defined by:

(Def. 9) $\mathrm{InvPairFunc} = \mathrm{PairFunc}^{-1}$.

Let us consider X, F, S_1, C_2. The functor $\mathrm{On}\, C_2$ yielding a covering of $\bigcup \mathrm{rng}\, S_1$ in F is defined by:

(Def. 10) For every natural number n holds $(\mathrm{On}\, C_2)(n) = C_2(\mathrm{pr1}(\mathrm{InvPairFunc})(n))(\mathrm{pr2}(\mathrm{InvPairFunc})(n))$.

The following propositions are true:

(6) Let k be an element of $\mathbb{N}$. Then there exists a natural number m such that for every sequence S_1 of subsets of X and for every covering C_2 of S_1 in F holds $(\sum_{\alpha=0}^{\kappa}(\mathrm{vol}(M, \mathrm{On}\, C_2))(\alpha))_{\kappa \in \mathbb{N}}(k) \leq (\sum_{\alpha=0}^{\kappa}(\mathrm{Volume}(M, C_2))(\alpha))_{\kappa \in \mathbb{N}}(m)$.

(7) $\inf \mathrm{Svc}(M, \bigcup \mathrm{rng}\, S_1) \leq \overline{\sum} \mathrm{Volume}(M, C_2)$.

(8) If $A \in F$, then $A, \emptyset_X$ followed by $\emptyset_X$ is a covering of A in F.

(9) Let X be a set, F be a field of subsets of X, M be a measure on F, and A be a set. If $A \in F$, then (the Caratheodory measure determined by $M)(A) \leq M(A)$.

(10) The Caratheodory measure determined by M is non-negative.

(11) (The Caratheodory measure determined by $M)(\emptyset) = 0$.

(12) If $A \subseteq B$, then (the Caratheodory measure determined by $M)(A) \leq$ (the Caratheodory measure determined by $M)(B)$.

(13) (The Caratheodory measure determined by $M)(\bigcup \mathrm{rng}\, S_1) \leq \overline{\sum}$((the Caratheodory measure determined by $M) \cdot S_1)$.

(14) The Caratheodory measure determined by M is a Caratheodor's measure on X.

Let X be a set, let F be a field of subsets of X, and let M be a measure on F. Then the Caratheodory measure determined by M is a Caratheodor's measure on X.

2. Hopf Extension Theorem

Let X be a set, let F be a field of subsets of X, and let M be a measure on F. We say that M is completely-additive if and only if:

(Def. 11) For every sequence F_1 of separated subsets of F such that $\bigcup \operatorname{rng} F_1 \in F$ holds $\overline{\sum}(M \cdot F_1) = M(\bigcup \operatorname{rng} F_1)$.

The following propositions are true:

(15) The partial unions of F_1 are a set sequence of F.

(16) The partial diff-unions of F_1 are a set sequence of F.

(17) Suppose $A \in F$. Then there exists a sequence F_1 of separated subsets of F such that $A = \bigcup \operatorname{rng} F_1$ and for every natural number n holds $F_1(n) \subseteq C_1(n)$.

(18) Suppose M is completely-additive. Let A be a set. If $A \in F$, then $M(A) =$ (the Caratheodory measure determined by M)(A).

In the sequel C is a Caratheodor's measure on X.

We now state three propositions:

(19) If for every subset B of X holds $C(B \cap A) + C(B \cap (X \setminus A)) \leq C(B)$, then $A \in \sigma$-Field(C).

(20) $F \subseteq \sigma$-Field(the Caratheodory measure determined by M).

(21) Let X be a set, F be a field of subsets of X, F_1 be a set sequence of F, and M be a function from F into $\overline{\mathbb{R}}$. Then $M \cdot F_1$ is a sequence of extended reals.

Let X be a set, let F be a field of subsets of X, let F_1 be a set sequence of F, and let g be a function from F into $\overline{\mathbb{R}}$. Then $g \cdot F_1$ is a sequence of extended reals.

One can prove the following proposition

(22) Let X be a set, S be a σ-field of subsets of X, S_2 be a sequence of subsets of S, and M be a function from S into $\overline{\mathbb{R}}$. Then $M \cdot S_2$ is a sequence of extended reals.

Let X be a set, let S be a σ-field of subsets of X, let S_2 be a sequence of subsets of S, and let g be a function from S into $\overline{\mathbb{R}}$. Then $g \cdot S_2$ is a sequence of extended reals.

Next we state several propositions:

(23) Let F, G be functions from $\mathbb{N}$ into $\overline{\mathbb{R}}$ and n be a natural number. Suppose that for every natural number m such that $m \leq n$ holds $F(m) \leq G(m)$. Then $(\operatorname{Ser} F)(n) \leq (\operatorname{Ser} G)(n)$.

(24) For all X, C and for every sequence s_1 of separated subsets of σ-Field(C) holds $\bigcup \operatorname{rng} s_1 \in \sigma$-Field$(C)$ and $C(\bigcup \operatorname{rng} s_1) = \sum(C \cdot s_1)$.

(25) For all X, C and for every sequence s_1 of subsets of σ-Field(C) holds $\bigcup s_1 \in \sigma$-Field(C).

(26) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and S_2 be a sequence of subsets of S. If S_2 is non-decreasing, then $\lim(M \cdot S_2) = M(\lim S_2)$.

(27) If F_1 is non-decreasing, then $M \cdot F_1$ is non-decreasing.

(28) If F_1 is descending, then $M \cdot F_1$ is non-increasing.

(29) Let X be a set, S be a σ-field of subsets of X, M be a σ-measure on S, and S_2 be a sequence of subsets of S. If S_2 is non-decreasing, then $M \cdot S_2$ is non-decreasing.

(30) Let X be a set, S be a σ-field of subsets of X, M be a σ-measure on S, and S_2 be a sequence of subsets of S. If S_2 is descending, then $M \cdot S_2$ is non-increasing.

(31) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and S_2 be a sequence of subsets of S. If S_2 is descending and $M(S_2(0)) < +\infty$, then $\lim(M \cdot S_2) = M(\lim S_2)$.

Let X be a set, let F be a field of subsets of X, let S be a σ-field of subsets of X, let m be a measure on F, and let M be a σ-measure on S. We say that M is an extension of m if and only if:

(Def. 12) For every set A such that $A \in F$ holds $M(A) = m(A)$.

The following four propositions are true:

(32) Let X be a non empty set, F be a field of subsets of X, and m be a measure on F. If there exists a σ-measure on $\sigma(F)$ which is an extension of m, then m is completely-additive.

(33) Let X be a non empty set, F be a field of subsets of X, and m be a measure on F. Suppose m is completely-additive. Then there exists a σ-measure M on $\sigma(F)$ such that M is an extension of m and $M = \sigma$-Meas(the Caratheodory measure determined by $m) \restriction \sigma(F)$.

(34) If for every n holds $M(F_1(n)) < +\infty$, then $M(($the partial unions of $F_1)(k)) < +\infty$.

(35) Let X be a non empty set, F be a field of subsets of X, and m be a measure on F. Suppose that

(i) m is completely-additive, and

(ii) there exists a set sequence A_1 of F such that for every natural number n holds $m(A_1(n)) < +\infty$ and $X = \bigcup \operatorname{rng} A_1$.

Let M be a σ-measure on $\sigma(F)$. Suppose M is an extension of m. Then $M = \sigma$-Meas(the Caratheodory measure determined by $m) \restriction \sigma(F)$.

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Received April 7, 2009