

Probability on Finite Set and Real-Valued Random Variables

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Summary. In the various branches of science, probability and randomness provide us with useful theoretical frameworks. The *Formalized Mathematics* has already published some articles concerning the probability: [23], [24], [25], and [30]. In order to apply those articles, we shall give some theorems concerning the probability and the real-valued random variables to prepare for further studies.

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The articles [12], [28], [3], [14], [1], [18], [27], [9], [29], [11], [4], [21], [10], [2], [5], [6], [20], [25], [24], [30], [7], [16], [17], [19], [8], [15], [26], [13], and [22] provide the notation and terminology for this paper.

1. PROBABILITY ON FINITE SET

One can prove the following four propositions:

- (1) Let X be a non empty set, S_1 be a σ -field of subsets of X , M be a σ -measure on S_1 , f be a partial function from X to $\overline{\mathbb{R}}$, E be an element of S_1 , and a be a real number. Suppose f is integrable on M and $E \subseteq \text{dom } f$ and $M(E) < +\infty$ and for every element x of X such that $x \in E$ holds $a \leq f(x)$. Then $\overline{\mathbb{R}}(a) \cdot M(E) \leq \int f \upharpoonright E \, dM$.
- (2) Let X be a non empty set, S_1 be a σ -field of subsets of X , M be a σ -measure on S_1 , f be a partial function from X to \mathbb{R} , E be an element of S_1 , and a be a real number. Suppose f is integrable on M and $E \subseteq \text{dom } f$ and $M(E) < +\infty$ and for every element x of X such that $x \in E$ holds $a \leq f(x)$. Then $\overline{\mathbb{R}}(a) \cdot M(E) \leq \int f \upharpoonright E \, dM$.

- (3) Let X be a non empty set, S_1 be a σ -field of subsets of X , M be a σ -measure on S_1 , f be a partial function from X to $\overline{\mathbb{R}}$, E be an element of S_1 , and a be a real number. Suppose f is integrable on M and $E \subseteq \text{dom } f$ and $M(E) < +\infty$ and for every element x of X such that $x \in E$ holds $f(x) \leq a$. Then $\int f|_E dM \leq \overline{\mathbb{R}}(a) \cdot M(E)$.
- (4) Let X be a non empty set, S_1 be a σ -field of subsets of X , M be a σ -measure on S_1 , f be a partial function from X to \mathbb{R} , E be an element of S_1 , and a be a real number. Suppose f is integrable on M and $E \subseteq \text{dom } f$ and $M(E) < +\infty$ and for every element x of X such that $x \in E$ holds $f(x) \leq a$. Then $\int f|_E dM \leq \overline{\mathbb{R}}(a) \cdot M(E)$.

2. RANDOM VARIABLES

For simplicity, we follow the rules: O is a non empty set, r is a real number, S is a σ -field of subsets of O , P is a probability on S , and E is a finite non empty set.

Let E be a non empty set. We introduce the trivial σ -field of E as a synonym of 2^E . Then the trivial σ -field of E is a σ -field of subsets of E .

Next we state a number of propositions:

- (5) Let O be a non empty finite set and f be a partial function from O to \mathbb{R} . Then there exists a finite sequence F of separated subsets of the trivial σ -field of O and there exists a finite sequence s of elements of $\text{dom } f$ such that
 $\text{dom } f = \bigcup \text{rng } F$ and $\text{dom } F = \text{dom } s$ and s is one-to-one and $\text{rng } s = \text{dom } f$ and $\text{len } s = \overline{\text{dom } f}$ and for every natural number k such that $k \in \text{dom } F$ holds $F(k) = \{s(k)\}$ and for every natural number n and for all elements x, y of O such that $n \in \text{dom } F$ and $x, y \in F(n)$ holds $f(x) = f(y)$.
- (6) Let O be a non empty finite set and f be a partial function from O to \mathbb{R} . Then
- (i) f is simple function in the trivial σ -field of O , and
 - (ii) $\text{dom } f$ is an element of the trivial σ -field of O .
- (7) Let O be a non empty finite set, M be a σ -measure on the trivial σ -field of O , and f be a partial function from O to \mathbb{R} . If $\text{dom } f \neq \emptyset$ and $M(\text{dom } f) < +\infty$, then f is integrable on M .
- (8) Let O be a non empty finite set and f be a partial function from O to \mathbb{R} . Then there exists an element X of the trivial σ -field of O such that $\text{dom } f = X$ and f is measurable on X .
- (9) Let O be a non empty finite set, M be a σ -measure on the trivial σ -field of O , f be a function from O into \mathbb{R} , x be a finite sequence of elements of $\overline{\mathbb{R}}$, and s be a finite sequence of elements of O . Suppose $M(O) < +\infty$ and

s is one-to-one and $\text{rng } s = O$ and $\text{len } s = \overline{O}$. Then there exists a finite sequence F of separated subsets of the trivial σ -field of O and there exists a finite sequence a of elements of \mathbb{R} such that

- (i) $\text{dom } f = \bigcup \text{rng } F$,
 - (ii) $\text{dom } a = \text{dom } s$,
 - (iii) $\text{dom } F = \text{dom } s$,
 - (iv) for every natural number k such that $k \in \text{dom } F$ holds $F(k) = \{s(k)\}$,
and
 - (v) for every natural number n and for all elements x, y of O such that $n \in \text{dom } F$ and $x, y \in F(n)$ holds $f(x) = f(y)$.
- (10) Let O be a non empty finite set, M be a σ -measure on the trivial σ -field of O , f be a function from O into \mathbb{R} , x be a finite sequence of elements of \mathbb{R} , and s be a finite sequence of elements of O . Suppose that
- (i) $M(O) < +\infty$,
 - (ii) $\text{len } x = \overline{O}$,
 - (iii) s is one-to-one,
 - (iv) $\text{rng } s = O$,
 - (v) $\text{len } s = \overline{O}$, and
 - (vi) for every natural number n such that $n \in \text{dom } x$ holds $x(n) = \mathbb{R}(f(s(n))) \cdot M(\{s(n)\})$.
- Then $\int f \, dM = \sum x$.
- (11) Let O be a non empty finite set, M be a σ -measure on the trivial σ -field of O , and f be a function from O into \mathbb{R} . Suppose $M(O) < +\infty$. Then there exists a finite sequence x of elements of \mathbb{R} and there exists a finite sequence s of elements of O such that
- (i) $\text{len } x = \overline{O}$,
 - (ii) s is one-to-one,
 - (iii) $\text{rng } s = O$,
 - (iv) $\text{len } s = \overline{O}$,
 - (v) for every natural number n such that $n \in \text{dom } x$ holds $x(n) = \mathbb{R}(f(s(n))) \cdot M(\{s(n)\})$, and
 - (vi) $\int f \, dM = \sum x$.
- (12) Let O be a non empty finite set, P be a probability on the trivial σ -field of O , f be a function from O into \mathbb{R} , x be a finite sequence of elements of \mathbb{R} , and s be a finite sequence of elements of O . Suppose that
- (i) $\text{len } x = \overline{O}$,
 - (ii) s is one-to-one,
 - (iii) $\text{rng } s = O$,
 - (iv) $\text{len } s = \overline{O}$, and
 - (v) for every natural number n such that $n \in \text{dom } x$ holds $x(n) = f(s(n)) \cdot P(\{s(n)\})$.

Then $\int f \, dP = \sum x$.

- (13) Let O be a non empty finite set, P be a probability on the trivial σ -field of O , and f be a function from O into \mathbb{R} . Then there exists a finite sequence F of elements of \mathbb{R} and there exists a finite sequence s of elements of O such that
- (i) $\text{len } F = \overline{O}$,
 - (ii) s is one-to-one,
 - (iii) $\text{rng } s = O$,
 - (iv) $\text{len } s = \overline{O}$,
 - (v) for every natural number n such that $n \in \text{dom } F$ holds $F(n) = f(s(n))$.
- $P(\{s(n)\})$, and
- (vi) $\int f \, dP = \sum F$.
- (14) Let E be a finite non empty set and A be a sequence of subsets of E . Suppose A is non-increasing. Then there exists an element N of \mathbb{N} such that for every element m of \mathbb{N} such that $N \leq m$ holds $A(N) = A(m)$.
- (15) Let E be a finite non empty set and A be a sequence of subsets of E . Suppose A is non-increasing. Then there exists an element N of \mathbb{N} such that for every element m of \mathbb{N} such that $N \leq m$ holds $\text{Intersection } A = A(m)$.
- (16) Let E be a finite non empty set and A be a sequence of subsets of E . Suppose A is non-decreasing. Then there exists an element N of \mathbb{N} such that for every element m of \mathbb{N} such that $N \leq m$ holds $A(N) = A(m)$.
- (17) Let E be a finite non empty set and A be a sequence of subsets of E . Suppose A is non-decreasing. Then there exists a natural number N such that for every natural number m such that $N \leq m$ holds $\bigcup A = A(m)$.

Let us consider E . The trivial probability of E yielding a probability on the trivial σ -field of E is defined as follows:

- (Def. 1) For every event A_1 of E holds (the trivial probability of E)(A_1) = $P(A_1)$.

Let us consider O, S . A function from O into \mathbb{R} is said to be a real-valued random variable of S if:

- (Def. 2) There exists an element X of S such that $X = O$ and it is measurable on X .

In the sequel f, g are real-valued random variables of S .

Next we state the proposition

- (18) $f + g$ is a real-valued random variable of S .

Let us consider O, S, f, g . Then $f + g$ is a real-valued random variable of S .

We now state the proposition

- (19) $f - g$ is a real-valued random variable of S .

Let us consider O, S, f, g . Then $f - g$ is a real-valued random variable of S .

Next we state the proposition

- (20) For every real number r holds $r f$ is a real-valued random variable of S .

Let us consider O, S, f and let r be a real number. Then rf is a real-valued random variable of S .

Next we state two propositions:

- (21) For all partial functions f, g from O to \mathbb{R} holds $\overline{\mathbb{R}}(f) \overline{\mathbb{R}}(g) = \overline{\mathbb{R}}(fg)$.
- (22) fg is a real-valued random variable of S .

Let us consider O, S, f, g . Then fg is a real-valued random variable of S .

Next we state two propositions:

- (23) For every real number r such that $0 \leq r$ and f is non-negative holds f^r is a real-valued random variable of S .
- (24) $|f|$ is a real-valued random variable of S .

Let us consider O, S, f . Then $|f|$ is a real-valued random variable of S .

We now state the proposition

- (25) For every real number r such that $0 \leq r$ holds $|f|^r$ is a real-valued random variable of S .

Let us consider O, S, f, P . We say that f is integrable on P if and only if:

(Def. 3) f is integrable on P2M P .

Let us consider O, S, P and let f be a real-valued random variable of S . Let us assume that f is integrable on P . The functor $E_P\{f\}$ yielding an element of \mathbb{R} is defined as follows:

(Def. 4) $E_P\{f\} = \int f \, d\text{P2M } P$.

One can prove the following propositions:

- (26) If f is integrable on P and g is integrable on P , then $E_P\{f + g\} = E_P\{f\} + E_P\{g\}$.
- (27) If f is integrable on P , then $E_P\{rf\} = r \cdot E_P\{f\}$.
- (28) If f is integrable on P and g is integrable on P , then $E_P\{f - g\} = E_P\{f\} - E_P\{g\}$.
- (29) For every non empty finite set O holds every function from O into \mathbb{R} is a real-valued random variable of the trivial σ -field of O .
- (30) Let O be a non empty finite set, P be a probability on the trivial σ -field of O , and X be a real-valued random variable of the trivial σ -field of O . Then X is integrable on P .
- (31) Let O be a non empty finite set, P be a probability on the trivial σ -field of O , X be a real-valued random variable of the trivial σ -field of O , F be a finite sequence of elements of \mathbb{R} , and s be a finite sequence of elements of O . Suppose that
 - (i) $\text{len } F = \overline{\overline{O}}$,
 - (ii) s is one-to-one,
 - (iii) $\text{rng } s = O$,
 - (iv) $\text{len } s = \overline{\overline{O}}$, and

- (v) for every natural number n such that $n \in \text{dom } F$ holds $F(n) = X(s(n)) \cdot P(\{s(n)\})$.
Then $E_P\{X\} = \sum F$.
- (32) Let O be a non empty finite set, P be a probability on the trivial σ -field of O , and X be a real-valued random variable of the trivial σ -field of O . Then there exists a finite sequence F of elements of \mathbb{R} and there exists a finite sequence s of elements of O such that
- (i) $\text{len } F = \overline{\overline{O}}$,
 - (ii) s is one-to-one,
 - (iii) $\text{rng } s = O$,
 - (iv) $\text{len } s = \overline{\overline{O}}$,
 - (v) for every natural number n such that $n \in \text{dom } F$ holds $F(n) = X(s(n)) \cdot P(\{s(n)\})$, and
 - (vi) $E_P\{X\} = \sum F$.
- (33) Let O be a non empty finite set, P be a probability on the trivial σ -field of O , and X be a real-valued random variable of the trivial σ -field of O . Then there exists a finite sequence F of elements of \mathbb{R} and there exists a finite sequence s of elements of O such that
- (i) $\text{len } F = \overline{\overline{O}}$,
 - (ii) s is one-to-one,
 - (iii) $\text{rng } s = O$,
 - (iv) $\text{len } s = \overline{\overline{O}}$,
 - (v) for every natural number n such that $n \in \text{dom } F$ holds $F(n) = X(s(n)) \cdot P(\{s(n)\})$, and
 - (vi) $E_P\{X\} = \sum F$.
- (34) Let O be a non empty finite set, X be a real-valued random variable of the trivial σ -field of O , G be a finite sequence of elements of \mathbb{R} , and s be a finite sequence of elements of O . Suppose $\text{len } G = \overline{\overline{O}}$ and s is one-to-one and $\text{rng } s = O$ and $\text{len } s = \overline{\overline{O}}$ and for every natural number n such that $n \in \text{dom } G$ holds $G(n) = X(s(n))$. Then $E_{\text{the trivial probability of } O}\{X\} = \sum_{\overline{\overline{O}}} G$.
- (35) Let O be a non empty finite set and X be a real-valued random variable of the trivial σ -field of O . Then there exists a finite sequence G of elements of \mathbb{R} and there exists a finite sequence s of elements of O such that
- (i) $\text{len } G = \overline{\overline{O}}$,
 - (ii) s is one-to-one,
 - (iii) $\text{rng } s = O$,
 - (iv) $\text{len } s = \overline{\overline{O}}$,
 - (v) for every natural number n such that $n \in \text{dom } G$ holds $G(n) = X(s(n))$, and

- (vi) $E_{\text{the trivial probability of } O\{X\}} = \frac{\sum G}{O}$.
- (36) Let X be a real-valued random variable of S . Suppose $0 < r$ and X is non-negative and X is integrable on P . Then $P(\{t \in O: r \leq X(t)\}) \leq \frac{E_P\{X\}}{r}$.

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