

## Complex Function Differentiability

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**Summary.** For a complex valued function defined on its domain in complex numbers the differentiability in a single point and on a subset of the domain is presented. The main elements of differential calculus are developed. The algebraic properties of differential complex functions are shown.

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The articles [17], [18], [3], [5], [4], [8], [2], [7], [11], [6], [16], [12], [19], [9], [10], [1], [14], [15], and [13] provide the notation and terminology for this paper.

For simplicity, we use the following convention:  $k, n, m$  denote elements of  $\mathbb{N}$ ,  $X$  denotes a set,  $s_1, s_2$  denote complex sequences,  $Y$  denotes a subset of  $\mathbb{C}$ ,  $f, f_1, f_2$  denote partial functions from  $\mathbb{C}$  to  $\mathbb{C}$ ,  $r$  denotes a real number,  $a, a_1, b, x, x_0, z, z_0$  denote complex numbers, and  $N_1$  denotes an increasing sequence of naturals.

Let  $I$  be a complex sequence. We say that  $I$  is convergent to 0 if and only if:

(Def. 1)  $I$  is non-zero and convergent and  $\lim I = 0$ .

We now state four propositions:

- (1) Let  $r_1$  be a sequence of real numbers and  $c_1$  be a complex sequence. If  $r_1 = c_1$  and  $r_1$  is convergent, then  $c_1$  is convergent.
- (2) If  $0 < r$  and for every  $n$  holds  $s_1(n) = \frac{1}{n+r}$ , then  $s_1$  is convergent.
- (3) If  $0 < r$  and for every  $n$  holds  $s_1(n) = \frac{1}{n+r}$ , then  $\lim s_1 = 0$ .
- (4) If for every  $n$  holds  $s_1(n) = \frac{1}{n+1}$ , then  $s_1$  is convergent and  $\lim s_1 = 0$ .

Let us observe that there exists a complex sequence which is convergent to 0.

Let us note that there exists a complex sequence which is constant.

Next we state four propositions:

- (5)  $s_1$  is constant iff for all  $n, m$  holds  $s_1(n) = s_1(m)$ .
- (6) For every  $n$  holds  $(s_1 \cdot N_1)(n) = s_1(N_1(n))$ .
- (7) If  $s_1$  is constant and  $s_2$  is a subsequence of  $s_1$ , then  $s_2$  is constant.
- (8) If  $s_1$  is constant and  $s_2$  is a subsequence of  $s_1$ , then  $s_1 = s_2$ .

Let  $s_3$  be a constant complex sequence. Note that every subsequence of  $s_3$  is constant.

In the sequel  $h$  is a convergent to 0 complex sequence and  $c$  is a constant complex sequence.

Let  $I$  be a partial function from  $\mathbb{C}$  to  $\mathbb{C}$ . We say that  $I$  is rest-like if and only if:

- (Def. 2)  $I$  is total and for every  $h$  holds  $h^{-1}(I \cdot h)$  is convergent and  $\lim(h^{-1}(I \cdot h)) = 0$ .

Let us mention that there exists a partial function from  $\mathbb{C}$  to  $\mathbb{C}$  which is rest-like.

A  $\mathbb{C}$ -rest is a rest-like partial function from  $\mathbb{C}$  to  $\mathbb{C}$ .

Let  $I$  be a partial function from  $\mathbb{C}$  to  $\mathbb{C}$ . We say that  $I$  is linear if and only if:

- (Def. 3)  $I$  is total and there exists  $a$  such that for every  $z$  holds  $I_z = a \cdot z$ .

One can check that there exists a partial function from  $\mathbb{C}$  to  $\mathbb{C}$  which is linear.

A  $\mathbb{C}$ -linear function is a linear partial function from  $\mathbb{C}$  to  $\mathbb{C}$ .

We adopt the following convention:  $R, R_1, R_2$  are  $\mathbb{C}$ -rests and  $L, L_1, L_2$  are  $\mathbb{C}$ -linear functions.

Let us consider  $L_1, L_2$ . Observe that  $L_1 + L_2$  is linear and  $L_1 - L_2$  is linear.

The following propositions are true:

- (9) For all  $L_1, L_2$  holds  $L_1 + L_2$  is a  $\mathbb{C}$ -linear function and  $L_1 - L_2$  is a  $\mathbb{C}$ -linear function.
- (10) For all  $a, L$  holds  $aL$  is a  $\mathbb{C}$ -linear function.
- (11) For all  $R_1, R_2$  holds  $R_1 + R_2$  is a  $\mathbb{C}$ -rest and  $R_1 - R_2$  is a  $\mathbb{C}$ -rest and  $R_1 R_2$  is a  $\mathbb{C}$ -rest.
- (12)  $aR$  is a  $\mathbb{C}$ -rest.
- (13)  $L_1 L_2$  is rest-like.
- (14)  $RL$  is a  $\mathbb{C}$ -rest and  $LR$  is a  $\mathbb{C}$ -rest.

Let  $z_0$  be a complex number. A subset of  $\mathbb{C}$  is called a neighbourhood of  $z_0$  if:

(Def. 4) There exists a real number  $g$  such that  $0 < g$  and  $\{y; y \text{ ranges over complex numbers: } |y - z_0| < g\} \subseteq \text{it}$ .

Next we state three propositions:

(15) For every real number  $g$  such that  $0 < g$  holds  $\{y; y \text{ ranges over complex numbers: } |y - z_0| < g\}$  is a neighbourhood of  $z_0$ .

(16) For every neighbourhood  $N$  of  $z_0$  holds  $z_0 \in N$ .

(17) Let  $z_0$  be a complex number and  $N_2, N_3$  be neighbourhoods of  $z_0$ . Then there exists a neighbourhood  $N$  of  $z_0$  such that  $N \subseteq N_2$  and  $N \subseteq N_3$ .

Let us consider  $f$  and let  $x_0$  be a complex number. We say that  $f$  is differentiable in  $x_0$  if and only if the condition (Def. 5) is satisfied.

(Def. 5) There exists a neighbourhood  $N$  of  $x_0$  such that  $N \subseteq \text{dom } f$  and there exist  $L, R$  such that for every complex number  $x$  such that  $x \in N$  holds  $f_x - f_{x_0} = L_{x-x_0} + R_{x-x_0}$ .

Let us consider  $f$  and let  $z_0$  be a complex number. Let us assume that  $f$  is differentiable in  $z_0$ . The functor  $f'(z_0)$  yielding a complex number is defined by the condition (Def. 6).

(Def. 6) There exists a neighbourhood  $N$  of  $z_0$  such that  $N \subseteq \text{dom } f$  and there exist  $L, R$  such that  $f'(z_0) = L_{1\mathbb{C}}$  and for every complex number  $z$  such that  $z \in N$  holds  $f_z - f_{z_0} = L_{z-z_0} + R_{z-z_0}$ .

Let us consider  $f, X$ . We say that  $f$  is differentiable on  $X$  if and only if:

(Def. 7)  $X \subseteq \text{dom } f$  and for every  $x$  such that  $x \in X$  holds  $f|X$  is differentiable in  $x$ .

We now state the proposition

(18) If  $f$  is differentiable on  $X$ , then  $X$  is a subset of  $\mathbb{C}$ .

Let  $X$  be a subset of  $\mathbb{C}$ . We say that  $X$  is closed if and only if:

(Def. 8) For every complex sequence  $s_3$  such that  $\text{rng } s_3 \subseteq X$  and  $s_3$  is convergent holds  $\lim s_3 \in X$ .

Let  $X$  be a subset of  $\mathbb{C}$ . We say that  $X$  is open if and only if:

(Def. 9)  $X^c$  is closed.

Next we state several propositions:

(19) Let  $X$  be a subset of  $\mathbb{C}$ . Suppose  $X$  is open. Let  $z_0$  be a complex number. If  $z_0 \in X$ , then there exists a neighbourhood  $N$  of  $z_0$  such that  $N \subseteq X$ .

(20) Let  $X$  be a subset of  $\mathbb{C}$ . Suppose  $X$  is open. Let  $z_0$  be a complex number. Suppose  $z_0 \in X$ . Then there exists a real number  $g$  such that  $\{y; y \text{ ranges over complex numbers: } |y - z_0| < g\} \subseteq X$ .

(21) Let  $X$  be a subset of  $\mathbb{C}$ . Suppose that for every complex number  $z_0$  such that  $z_0 \in X$  there exists a neighbourhood  $N$  of  $z_0$  such that  $N \subseteq X$ . Then  $X$  is open.

- (22) Let  $X$  be a subset of  $\mathbb{C}$ . Then  $X$  is open if and only if for every complex number  $x$  such that  $x \in X$  there exists a neighbourhood  $N$  of  $x$  such that  $N \subseteq X$ .
- (23) Let  $X$  be a subset of  $\mathbb{C}$ ,  $z_0$  be an element of  $\mathbb{C}$ , and  $r$  be an element of  $\mathbb{R}$ . If  $X = \{y; y \text{ ranges over complex numbers: } |y - z_0| < r\}$ , then  $X$  is open.
- (24) Let  $X$  be a subset of  $\mathbb{C}$ ,  $z_0$  be an element of  $\mathbb{C}$ , and  $r$  be an element of  $\mathbb{R}$ . If  $X = \{y; y \text{ ranges over complex numbers: } |y - z_0| \leq r\}$ , then  $X$  is closed.

Let us note that there exists a subset of  $\mathbb{C}$  which is open.

In the sequel  $Z$  denotes an open subset of  $\mathbb{C}$ .

Next we state two propositions:

- (25)  $f$  is differentiable on  $Z$  iff  $Z \subseteq \text{dom } f$  and for every  $x$  such that  $x \in Z$  holds  $f$  is differentiable in  $x$ .
- (26) If  $f$  is differentiable on  $Y$ , then  $Y$  is open.

Let us consider  $f, X$ . Let us assume that  $f$  is differentiable on  $X$ . The functor  $f'_{\uparrow X}$  yielding a partial function from  $\mathbb{C}$  to  $\mathbb{C}$  is defined by:

(Def. 10)  $\text{dom}(f'_{\uparrow X}) = X$  and for every  $x$  such that  $x \in X$  holds  $(f'_{\uparrow X})_x = f'(x)$ .

The following propositions are true:

- (27) Let given  $f, Z$ . Suppose  $Z \subseteq \text{dom } f$  and there exists  $a_1$  such that  $\text{rng } f = \{a_1\}$ . Then  $f$  is differentiable on  $Z$  and for every  $x$  such that  $x \in Z$  holds  $(f'_{\uparrow Z})_x = 0_{\mathbb{C}}$ .
- (28) If  $s_1$  is non-zero, then  $s_1 \uparrow k$  is non-zero.

Let us consider  $h, n$ . Note that  $h \uparrow n$  is convergent to 0.

Let us consider  $c, n$ . Note that  $c \uparrow n$  is constant.

Next we state a number of propositions:

- (29)  $(s_1 + s_2) \uparrow k = s_1 \uparrow k + s_2 \uparrow k$ .
- (30)  $(s_1 - s_2) \uparrow k = s_1 \uparrow k - s_2 \uparrow k$ .
- (31)  $s_1^{-1} \uparrow k = (s_1 \uparrow k)^{-1}$ .
- (32)  $(s_1 s_2) \uparrow k = (s_1 \uparrow k) (s_2 \uparrow k)$ .
- (33) Let  $x_0$  be a complex number and  $N$  be a neighbourhood of  $x_0$ . Suppose  $f$  is differentiable in  $x_0$  and  $N \subseteq \text{dom } f$ . Let given  $h, c$ . Suppose  $\text{rng } c = \{x_0\}$  and  $\text{rng}(h + c) \subseteq N$ . Then  $h^{-1}(f \cdot (h + c) - f \cdot c)$  is convergent and  $f'(x_0) = \lim(h^{-1}(f \cdot (h + c) - f \cdot c))$ .
- (34) Let given  $f_1, f_2, x_0$ . Suppose  $f_1$  is differentiable in  $x_0$  and  $f_2$  is differentiable in  $x_0$ . Then  $f_1 + f_2$  is differentiable in  $x_0$  and  $(f_1 + f_2)'(x_0) = f_1'(x_0) + f_2'(x_0)$ .
- (35) Let given  $f_1, f_2, x_0$ . Suppose  $f_1$  is differentiable in  $x_0$  and  $f_2$  is differentiable in  $x_0$ . Then  $f_1 - f_2$  is differentiable in  $x_0$  and  $(f_1 - f_2)'(x_0) = f_1'(x_0) - f_2'(x_0)$ .

- (36) For all  $a, f, x_0$  such that  $f$  is differentiable in  $x_0$  holds  $a f$  is differentiable in  $x_0$  and  $(a f)'(x_0) = a \cdot f'(x_0)$ .
- (37) Let given  $f_1, f_2, x_0$ . Suppose  $f_1$  is differentiable in  $x_0$  and  $f_2$  is differentiable in  $x_0$ . Then  $f_1 f_2$  is differentiable in  $x_0$  and  $(f_1 f_2)'(x_0) = (f_2)_{x_0} \cdot f_1'(x_0) + (f_1)_{x_0} \cdot f_2'(x_0)$ .
- (38) For all  $f, Z$  such that  $Z \subseteq \text{dom } f$  and  $f|_Z = \text{id}_Z$  holds  $f$  is differentiable on  $Z$  and for every  $x$  such that  $x \in Z$  holds  $(f|_Z)'_x = 1_{\mathbb{C}}$ .
- (39) Let given  $f_1, f_2, Z$ . Suppose  $Z \subseteq \text{dom}(f_1 + f_2)$  and  $f_1$  is differentiable on  $Z$  and  $f_2$  is differentiable on  $Z$ . Then  $f_1 + f_2$  is differentiable on  $Z$  and for every  $x$  such that  $x \in Z$  holds  $((f_1 + f_2)'|_Z)_x = f_1'(x) + f_2'(x)$ .
- (40) Let given  $f_1, f_2, Z$ . Suppose  $Z \subseteq \text{dom}(f_1 - f_2)$  and  $f_1$  is differentiable on  $Z$  and  $f_2$  is differentiable on  $Z$ . Then  $f_1 - f_2$  is differentiable on  $Z$  and for every  $x$  such that  $x \in Z$  holds  $((f_1 - f_2)'|_Z)_x = f_1'(x) - f_2'(x)$ .
- (41) Let given  $a, f, Z$ . Suppose  $Z \subseteq \text{dom}(a f)$  and  $f$  is differentiable on  $Z$ . Then  $a f$  is differentiable on  $Z$  and for every  $x$  such that  $x \in Z$  holds  $((a f)'|_Z)_x = a \cdot f'(x)$ .
- (42) Let given  $f_1, f_2, Z$ . Suppose  $Z \subseteq \text{dom}(f_1 f_2)$  and  $f_1$  is differentiable on  $Z$  and  $f_2$  is differentiable on  $Z$ . Then  $f_1 f_2$  is differentiable on  $Z$  and for every  $x$  such that  $x \in Z$  holds  $((f_1 f_2)'|_Z)_x = (f_2)_x \cdot f_1'(x) + (f_1)_x \cdot f_2'(x)$ .
- (43) If  $Z \subseteq \text{dom } f$  and  $f$  is a constant on  $Z$ , then  $f$  is differentiable on  $Z$  and for every  $x$  such that  $x \in Z$  holds  $(f|_Z)'_x = 0_{\mathbb{C}}$ .
- (44) Suppose  $Z \subseteq \text{dom } f$  and for every  $x$  such that  $x \in Z$  holds  $f_x = a \cdot x + b$ . Then  $f$  is differentiable on  $Z$  and for every  $x$  such that  $x \in Z$  holds  $(f|_Z)'_x = a$ .
- (45) For every complex number  $x_0$  such that  $f$  is differentiable in  $x_0$  holds  $f$  is continuous in  $x_0$ .
- (46) If  $f$  is differentiable on  $X$ , then  $f$  is continuous on  $X$ .
- (47) If  $f$  is differentiable on  $X$  and  $Z \subseteq X$ , then  $f$  is differentiable on  $Z$ .
- (48) If  $s_1$  is convergent, then  $|s_1|$  is convergent.
- (49) If  $f$  is differentiable in  $x_0$ , then there exists  $R$  such that  $R_{0_{\mathbb{C}}} = 0_{\mathbb{C}}$  and  $R$  is continuous in  $0_{\mathbb{C}}$ .

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