FORMALIZED MATHEMATICS
Vol. **17**, No. **1**, Pages 43–60, **2009**
DOI: 10.2478/v10037-009-0005-y

Arithmetic Operations on Functions from Sets into Functional Sets

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Summary. In this paper we introduce sets containing number-valued functions. Different arithmetic operations on maps between any set and such functional sets are later defined.

MML identifier: `VALUED_2`, version: `7.11.01 4.117.1046`

The notation and terminology used here are introduced in the following papers: [4], [9], [10], [2], [11], [6], [3], [1], [8], [5], and [7].

1. Functional sets

In this paper x, X, X_1, X_2 are sets.

Let Y be a functional set. The functor $\mathrm{DOMS}(Y)$ is defined by:

(Def. 1) $\mathrm{DOMS}(Y) = \bigcup\{\mathrm{dom}\, f : f$ ranges over elements of $Y\}$.

Let us consider X. We say that X is complex-functions-membered if and only if:

(Def. 2) If $x \in X$, then x is a complex-valued function.

Let us consider X. We say that X is extended-real-functions-membered if and only if:

(Def. 3) If $x \in X$, then x is an extended real-valued function.

Let us consider X. We say that X is real-functions-membered if and only if:

[1]The article was written while the author visited Shinshu University, Nagano, Japan.

ISSN 1426–2630(p), 1898-9934(e)

(Def. 4) If $x \in X$, then x is a real-valued function.

Let us consider X. We say that X is rational-functions-membered if and only if:

(Def. 5) If $x \in X$, then x is a rational-valued function.

Let us consider X. We say that X is integer-functions-membered if and only if:

(Def. 6) If $x \in X$, then x is an integer-valued function.

Let us consider X. We say that X is natural-functions-membered if and only if:

(Def. 7) If $x \in X$, then x is a natural-valued function.

One can check the following observations:

* every set which is natural-functions-membered is also integer-functions-membered,
* every set which is integer-functions-membered is also rational-functions-membered,
* every set which is rational-functions-membered is also real-functions-membered,
* every set which is real-functions-membered is also complex-functions-membered, and
* every set which is real-functions-membered is also extended-real-functions-membered.

Let us mention that every set which is empty is also natural-functions-membered.

Let f be a complex-valued function. Observe that $\{f\}$ is complex-functions-membered.

One can verify that every set which is complex-functions-membered is also functional and every set which is extended-real-functions-membered is also functional.

One can verify that there exists a set which is natural-functions-membered and non empty.

Let X be a complex-functions-membered set. One can verify that every subset of X is complex-functions-membered.

Let X be an extended-real-functions-membered set. Note that every subset of X is extended-real-functions-membered.

Let X be a real-functions-membered set. Note that every subset of X is real-functions-membered.

Let X be a rational-functions-membered set. Observe that every subset of X is rational-functions-membered.

Let X be an integer-functions-membered set. Note that every subset of X is integer-functions-membered.

Let X be a natural-functions-membered set. Observe that every subset of X is natural-functions-membered.

Let D be a set. The functor $\mathbb{C}$-PFuncs D yields a set and is defined by:

(Def. 8) For every set f holds $f \in \mathbb{C}$-PFuncs D iff f is a partial function from D to $\mathbb{C}$.

Let D be a set. The functor $\mathbb{C}$-Funcs D yielding a set is defined by:

(Def. 9) For every set f holds $f \in \mathbb{C}$-Funcs D iff f is a function from D into $\mathbb{C}$.

Let D be a set. The functor $\overline{\mathbb{R}}$-PFuncs D yields a set and is defined by:

(Def. 10) For every set f holds $f \in \overline{\mathbb{R}}$-PFuncs D iff f is a partial function from D to $\overline{\mathbb{R}}$.

Let D be a set. The functor $\overline{\mathbb{R}}$-Funcs D yields a set and is defined as follows:

(Def. 11) For every set f holds $f \in \overline{\mathbb{R}}$-Funcs D iff f is a function from D into $\overline{\mathbb{R}}$.

Let D be a set. The functor $\mathbb{R}$-PFuncs D yielding a set is defined by:

(Def. 12) For every set f holds $f \in \mathbb{R}$-PFuncs D iff f is a partial function from D to $\mathbb{R}$.

Let D be a set. The functor $\mathbb{R}$-Funcs D yielding a set is defined by:

(Def. 13) For every set f holds $f \in \mathbb{R}$-Funcs D iff f is a function from D into $\mathbb{R}$.

Let D be a set. The functor $\mathbb{Q}$-PFuncs D yields a set and is defined as follows:

(Def. 14) For every set f holds $f \in \mathbb{Q}$-PFuncs D iff f is a partial function from D to $\mathbb{Q}$.

Let D be a set. The functor $\mathbb{Q}$-Funcs D yields a set and is defined by:

(Def. 15) For every set f holds $f \in \mathbb{Q}$-Funcs D iff f is a function from D into $\mathbb{Q}$.

Let D be a set. The functor $\mathbb{Z}$-PFuncs D yielding a set is defined by:

(Def. 16) For every set f holds $f \in \mathbb{Z}$-PFuncs D iff f is a partial function from D to $\mathbb{Z}$.

Let D be a set. The functor $\mathbb{Z}$-Funcs D yields a set and is defined as follows:

(Def. 17) For every set f holds $f \in \mathbb{Z}$-Funcs D iff f is a function from D into $\mathbb{Z}$.

Let D be a set. The functor $\mathbb{N}$-PFuncs D yields a set and is defined by:

(Def. 18) For every set f holds $f \in \mathbb{N}$-PFuncs D iff f is a partial function from D to $\mathbb{N}$.

Let D be a set. The functor $\mathbb{N}$-Funcs D yielding a set is defined by:

(Def. 19) For every set f holds $f \in \mathbb{N}$-Funcs D iff f is a function from D into $\mathbb{N}$.

The following propositions are true:

(1) $\mathbb{C}$-Funcs X is a subset of $\mathbb{C}$-PFuncs X.

(2) $\overline{\mathbb{R}}$-Funcs X is a subset of $\overline{\mathbb{R}}$-PFuncs X.

(3) $\mathbb{R}$-Funcs X is a subset of $\mathbb{R}$-PFuncs X.

(4) $\mathbb{Q}$-Funcs X is a subset of $\mathbb{Q}$-PFuncs X.

(5) $\mathbb{Z}$-Funcs X is a subset of $\mathbb{Z}$-PFuncs X.

(6) $\mathbb{N}$-Funcs X is a subset of $\mathbb{N}$-PFuncs X.

Let us consider X. One can verify the following observations:

* $\mathbb{C}$-PFuncs X is complex-functions-membered,
* $\mathbb{C}$-Funcs X is complex-functions-membered,
* $\overline{\mathbb{R}}$-PFuncs X is extended-real-functions-membered,
* $\overline{\mathbb{R}}$-Funcs X is extended-real-functions-membered,
* $\mathbb{R}$-PFuncs X is real-functions-membered,
* $\mathbb{R}$-Funcs X is real-functions-membered,
* $\mathbb{Q}$-PFuncs X is rational-functions-membered,
* $\mathbb{Q}$-Funcs X is rational-functions-membered,
* $\mathbb{Z}$-PFuncs X is integer-functions-membered,
* $\mathbb{Z}$-Funcs X is integer-functions-membered,
* $\mathbb{N}$-PFuncs X is natural-functions-membered, and
* $\mathbb{N}$-Funcs X is natural-functions-membered.

Let X be a complex-functions-membered set. Observe that every element of X is complex-valued.

Let X be an extended-real-functions-membered set. One can check that every element of X is extended real-valued.

Let X be a real-functions-membered set. One can check that every element of X is real-valued.

Let X be a rational-functions-membered set. One can check that every element of X is rational-valued.

Let X be an integer-functions-membered set. Observe that every element of X is integer-valued.

Let X be a natural-functions-membered set. Observe that every element of X is natural-valued.

Let X, x be sets, let Y be a complex-functions-membered set, and let f be a partial function from X to Y. Observe that $f(x)$ is function-like and relation-like.

Let X, x be sets, let Y be an extended-real-functions-membered set, and let f be a partial function from X to Y. Observe that $f(x)$ is function-like and relation-like.

Let us consider X, x, let Y be a complex-functions-membered set, and let f be a partial function from X to Y. One can check that $f(x)$ is complex-valued.

Let us consider X, x, let Y be an extended-real-functions-membered set, and let f be a partial function from X to Y. One can verify that $f(x)$ is extended real-valued.

Let us consider X, x, let Y be a real-functions-membered set, and let f be a partial function from X to Y. Note that $f(x)$ is real-valued.

Let us consider X, x, let Y be a rational-functions-membered set, and let f be a partial function from X to Y. Note that $f(x)$ is rational-valued.

Let us consider X, x, let Y be an integer-functions-membered set, and let f be a partial function from X to Y. Note that $f(x)$ is integer-valued.

Let us consider X, x, let Y be a natural-functions-membered set, and let f be a partial function from X to Y. One can check that $f(x)$ is natural-valued.

Let us consider X and let Y be a complex-membered set. One can check that $X \dot{\rightarrow} Y$ is complex-functions-membered.

Let us consider X and let Y be an extended real-membered set. Observe that $X \dot{\rightarrow} Y$ is extended-real-functions-membered.

Let us consider X and let Y be a real-membered set. Observe that $X \dot{\rightarrow} Y$ is real-functions-membered.

Let us consider X and let Y be a rational-membered set. Observe that $X \dot{\rightarrow} Y$ is rational-functions-membered.

Let us consider X and let Y be an integer-membered set. Observe that $X \dot{\rightarrow} Y$ is integer-functions-membered.

Let us consider X and let Y be a natural-membered set. One can verify that $X \dot{\rightarrow} Y$ is natural-functions-membered.

Let us consider X and let Y be a complex-membered set. Note that Y^X is complex-functions-membered.

Let us consider X and let Y be an extended real-membered set. Note that Y^X is extended-real-functions-membered.

Let us consider X and let Y be a real-membered set. Note that Y^X is real-functions-membered.

Let us consider X and let Y be a rational-membered set. Note that Y^X is rational-functions-membered.

Let us consider X and let Y be an integer-membered set. Note that Y^X is integer-functions-membered.

Let us consider X and let Y be a natural-membered set. One can check that Y^X is natural-functions-membered.

Let R be a binary relation. We say that R is complex-functions-valued if and only if:

(Def. 20) rng R is complex-functions-membered.

We say that R is extended-real-functions-valued if and only if:

(Def. 21) rng R is extended-real-functions-membered.

We say that R is real-functions-valued if and only if:

(Def. 22) rng R is real-functions-membered.

We say that R is rational-functions-valued if and only if:

(Def. 23) rng R is rational-functions-membered.

We say that R is integer-functions-valued if and only if:

(Def. 24) rng R is integer-functions-membered.

We say that R is natural-functions-valued if and only if:

(Def. 25) rng R is natural-functions-membered.

Let f be a function. Let us observe that f is complex-functions-valued if and only if:

(Def. 26) For every set x such that $x \in \operatorname{dom} f$ holds $f(x)$ is a complex-valued function.

Let us observe that f is extended-real-functions-valued if and only if:

(Def. 27) For every set x such that $x \in \operatorname{dom} f$ holds $f(x)$ is an extended real-valued function.

Let us observe that f is real-functions-valued if and only if:

(Def. 28) For every set x such that $x \in \operatorname{dom} f$ holds $f(x)$ is a real-valued function.

Let us observe that f is rational-functions-valued if and only if:

(Def. 29) For every set x such that $x \in \operatorname{dom} f$ holds $f(x)$ is a rational-valued function.

Let us observe that f is integer-functions-valued if and only if:

(Def. 30) For every set x such that $x \in \operatorname{dom} f$ holds $f(x)$ is an integer-valued function.

Let us observe that f is natural-functions-valued if and only if:

(Def. 31) For every set x such that $x \in \operatorname{dom} f$ holds $f(x)$ is a natural-valued function.

One can verify the following observations:

* every binary relation which is natural-functions-valued is also integer-functions-valued,
* every binary relation which is integer-functions-valued is also rational-functions-valued,
* every binary relation which is rational-functions-valued is also real-functions-valued,
* every binary relation which is real-functions-valued is also extended-real-functions-valued, and
* every binary relation which is real-functions-valued is also complex-functions-valued.

Let us note that every binary relation which is empty is also natural-functions-valued.

Let us mention that there exists a function which is natural-functions-valued.

Let R be a complex-functions-valued binary relation. Note that rng R is complex-functions-membered.

Let R be an extended-real-functions-valued binary relation. Observe that rng R is extended-real-functions-membered.

Let R be a real-functions-valued binary relation. Note that $\operatorname{rng} R$ is real-functions-membered.

Let R be a rational-functions-valued binary relation. Observe that $\operatorname{rng} R$ is rational-functions-membered.

Let R be an integer-functions-valued binary relation. One can verify that $\operatorname{rng} R$ is integer-functions-membered.

Let R be a natural-functions-valued binary relation. One can check that $\operatorname{rng} R$ is natural-functions-membered.

Let us consider X and let Y be a complex-functions-membered set. Observe that every partial function from X to Y is complex-functions-valued.

Let us consider X and let Y be an extended-real-functions-membered set. One can check that every partial function from X to Y is extended-real-functions-valued.

Let us consider X and let Y be a real-functions-membered set. One can check that every partial function from X to Y is real-functions-valued.

Let us consider X and let Y be a rational-functions-membered set. Observe that every partial function from X to Y is rational-functions-valued.

Let us consider X and let Y be an integer-functions-membered set. Observe that every partial function from X to Y is integer-functions-valued.

Let us consider X and let Y be a natural-functions-membered set. Note that every partial function from X to Y is natural-functions-valued.

Let f be a complex-functions-valued function and let us consider x. Note that $f(x)$ is function-like and relation-like.

Let f be an extended-real-functions-valued function and let us consider x. Observe that $f(x)$ is function-like and relation-like.

Let f be a complex-functions-valued function and let us consider x. One can verify that $f(x)$ is complex-valued.

Let f be an extended-real-functions-valued function and let us consider x. Note that $f(x)$ is extended real-valued.

Let f be a real-functions-valued function and let us consider x. One can verify that $f(x)$ is real-valued.

Let f be a rational-functions-valued function and let us consider x. Observe that $f(x)$ is rational-valued.

Let f be an integer-functions-valued function and let us consider x. Note that $f(x)$ is integer-valued.

Let f be a natural-functions-valued function and let us consider x. One can check that $f(x)$ is natural-valued.

2. Operations

For simplicity, we adopt the following rules: Y, Y_1, Y_2 are complex-functions-membered sets, c, c_1, c_2 are complex numbers, f is a partial function from X

to Y, f_1 is a partial function from X_1 to Y_1, f_2 is a partial function from X_2 to Y_2, and g, h, k are complex-valued functions.

We now state a number of propositions:

(7) If $g \neq \emptyset$ and $g + c_1 = g + c_2$, then $c_1 = c_2$.

(8) If $g \neq \emptyset$ and $g - c_1 = g - c_2$, then $c_1 = c_2$.

(9) If $g \neq \emptyset$ and g is non-empty and $g\, c_1 = g\, c_2$, then $c_1 = c_2$.

(10) $-(g + c) = -g - c$.

(11) $-(g - c) = -g + c$.

(12) $(g + c_1) + c_2 = g + (c_1 + c_2)$.

(13) $(g + c_1) - c_2 = g + (c_1 - c_2)$.

(14) $(g - c_1) + c_2 = g - (c_1 - c_2)$.

(15) $g - c_1 - c_2 = g - (c_1 + c_2)$.

(16) $g\, c_1\, c_2 = g\, (c_1 \cdot c_2)$.

(17) $-(g + h) = -g - h$.

(18) $g - h = -(h - g)$.

(19) $(g\, h)/k = g\, (h/k)$.

(20) $(g/h)\, k = (g\, k)/h$.

(21) $g/h/k = g/(h\, k)$.

(22) $c\, -g = (-c)\, g$.

(23) $c\, -g = -c\, g$.

(24) $(-c)\, g = -c\, g$.

(25) $-g\, h = (-g)\, h$.

(26) $-g/h = (-g)/h$.

(27) $-g/h = g/-h$.

Let f be a complex-valued function and let c be a complex number. The functor f/c yields a function and is defined as follows:

(Def. 32) $f/c = \frac{1}{c}\, f$.

Let f be a complex-valued function and let c be a complex number. Note that f/c is complex-valued.

Let f be a real-valued function and let r be a real number. Note that f/r is real-valued.

Let f be a rational-valued function and let r be a rational number. One can check that f/r is rational-valued.

Let f be a complex-valued finite sequence and let c be a complex number. One can check that f/c is finite sequence-like.

The following propositions are true:

(28) $\mathrm{dom}(g/c) = \mathrm{dom}\, g$.

(29) $(g/c)(x) = \frac{g(x)}{c}$.

(30) $(-g)/c = -g/c$.

(31) $g/-c = -g/c$.

(32) $g/-c = (-g)/c$.

(33) If $g \neq \emptyset$ and g is non-empty and $g/c_1 = g/c_2$, then $c_1 = c_2$.

(34) $(g\,c_1)/c_2 = g\,\frac{c_1}{c_2}$.

(35) $(g/c_1)\,c_2 = (g\,c_2)/c_1$.

(36) $g/c_1/c_2 = g/(c_1 \cdot c_2)$.

(37) $(g+h)/c = g/c + h/c$.

(38) $(g-h)/c = g/c - h/c$.

(39) $(g\,h)/c = g\,(h/c)$.

(40) $(g/h)/c = g/(h\,c)$.

Let us consider X, let Y be a complex-functions-membered set, and let f be a partial function from X to Y. The functor $-f$ yields a function and is defined by:

(Def. 33) $\operatorname{dom}(-f) = \operatorname{dom} f$ and for every set x such that $x \in \operatorname{dom}(-f)$ holds $(-f)(x) = -f(x)$.

Let us consider X, let Y be a complex-functions-membered set, and let f be a partial function from X to Y. Then $-f$ is a partial function from X to $\mathbb{C}$-PFuncs DOMS(Y).

Let us consider X, let Y be a real-functions-membered set, and let f be a partial function from X to Y. Then $-f$ is a partial function from X to $\mathbb{R}$-PFuncs DOMS(Y).

Let us consider X, let Y be a rational-functions-membered set, and let f be a partial function from X to Y. Then $-f$ is a partial function from X to $\mathbb{Q}$-PFuncs DOMS(Y).

Let us consider X, let Y be an integer-functions-membered set, and let f be a partial function from X to Y. Then $-f$ is a partial function from X to $\mathbb{Z}$-PFuncs DOMS(Y).

Let Y be a complex-functions-membered set and let f be a finite sequence of elements of Y. One can check that $-f$ is finite sequence-like.

We now state two propositions:

(41) $--f = f$.

(42) If $-f_1 = -f_2$, then $f_1 = f_2$.

Let X be a complex-functions-membered set, let Y be a set, and let f be a partial function from X to Y. The functor $f \circ -$ yielding a function is defined as follows:

(Def. 34) $\operatorname{dom}(f \circ -) = \operatorname{dom} f$ and for every complex-valued function x such that $x \in \operatorname{dom}(f \circ -)$ holds $(f \circ -)(x) = f(-x)$.

Let us consider X, let Y be a complex-functions-membered set, and let f be a partial function from X to Y. The functor ${}^{1}/f$ yields a function and is defined as follows:

(Def. 35) $\operatorname{dom}{}^{1}/f = \operatorname{dom} f$ and for every set x such that $x \in \operatorname{dom}{}^{1}/f$ holds $({}^{1}/f)(x) = f(x)^{-1}$.

Let us consider X, let Y be a complex-functions-membered set, and let f be a partial function from X to Y. Then ${}^{1}/f$ is a partial function from X to $\mathbb{C}$-PFuncs DOMS(Y).

Let us consider X, let Y be a real-functions-membered set, and let f be a partial function from X to Y. Then ${}^{1}/f$ is a partial function from X to $\mathbb{R}$-PFuncs DOMS(Y).

Let us consider X, let Y be a rational-functions-membered set, and let f be a partial function from X to Y. Then ${}^{1}/f$ is a partial function from X to $\mathbb{Q}$-PFuncs DOMS(Y).

Let Y be a complex-functions-membered set and let f be a finite sequence of elements of Y. Note that ${}^{1}/f$ is finite sequence-like.

The following proposition is true

(43) ${}^{1}/{}^{1}/f = f$.

Let us consider X, let Y be a complex-functions-membered set, and let f be a partial function from X to Y. The functor $|f|$ yields a function and is defined by:

(Def. 36) $\operatorname{dom}|f| = \operatorname{dom} f$ and for every set x such that $x \in \operatorname{dom}|f|$ holds $|f|(x) = |f(x)|$.

Let us consider X, let Y be a complex-functions-membered set, and let f be a partial function from X to Y. Then $|f|$ is a partial function from X to $\mathbb{C}$-PFuncs DOMS(Y).

Let us consider X, let Y be a real-functions-membered set, and let f be a partial function from X to Y. Then $|f|$ is a partial function from X to $\mathbb{R}$-PFuncs DOMS(Y).

Let us consider X, let Y be a rational-functions-membered set, and let f be a partial function from X to Y. Then $|f|$ is a partial function from X to $\mathbb{Q}$-PFuncs DOMS(Y).

Let us consider X, let Y be an integer-functions-membered set, and let f be a partial function from X to Y. Then $|f|$ is a partial function from X to $\mathbb{N}$-PFuncs DOMS(Y).

Let Y be a complex-functions-membered set and let f be a finite sequence of elements of Y. Note that $|f|$ is finite sequence-like.

We now state the proposition

(44) $||f|| = |f|$.

Let us consider X, let Y be a complex-functions-membered set, let f be a partial function from X to Y, and let c be a complex number. The functor $f + c$

yields a function and is defined by:

(Def. 37) $\operatorname{dom}(f + c) = \operatorname{dom} f$ and for every set x such that $x \in \operatorname{dom}(f + c)$ holds $(f + c)(x) = c + f(x)$.

Let us consider X, let Y be a complex-functions-membered set, let f be a partial function from X to Y, and let c be a complex number. Then $f + c$ is a partial function from X to $\mathbb{C}$-PFuncs DOMS(Y).

Let us consider X, let Y be a real-functions-membered set, let f be a partial function from X to Y, and let c be a real number. Then $f+c$ is a partial function from X to $\mathbb{R}$-PFuncs DOMS(Y).

Let us consider X, let Y be a rational-functions-membered set, let f be a partial function from X to Y, and let c be a rational number. Then $f + c$ is a partial function from X to $\mathbb{Q}$-PFuncs DOMS(Y).

Let us consider X, let Y be an integer-functions-membered set, let f be a partial function from X to Y, and let c be an integer number. Then $f + c$ is a partial function from X to $\mathbb{Z}$-PFuncs DOMS(Y).

Let us consider X, let Y be a natural-functions-membered set, let f be a partial function from X to Y, and let c be a natural number. Then $f + c$ is a partial function from X to $\mathbb{N}$-PFuncs DOMS(Y).

One can prove the following propositions:

(45) $f + c_1 + c_2 = f + (c_1 + c_2)$.

(46) If $f \neq \emptyset$ and f is non-empty and $f + c_1 = f + c_2$, then $c_1 = c_2$.

Let us consider X, let Y be a complex-functions-membered set, let f be a partial function from X to Y, and let c be a complex number. The functor $f - c$ yields a function and is defined as follows:

(Def. 38) $f - c = f + -c$.

We now state two propositions:

(47) $\operatorname{dom}(f - c) = \operatorname{dom} f$.

(48) If $x \in \operatorname{dom}(f - c)$, then $(f - c)(x) = f(x) - c$.

Let us consider X, let Y be a complex-functions-membered set, let f be a partial function from X to Y, and let c be a complex number. Then $f - c$ is a partial function from X to $\mathbb{C}$-PFuncs DOMS(Y).

Let us consider X, let Y be a real-functions-membered set, let f be a partial function from X to Y, and let c be a real number. Then $f-c$ is a partial function from X to $\mathbb{R}$-PFuncs DOMS(Y).

Let us consider X, let Y be a rational-functions-membered set, let f be a partial function from X to Y, and let c be a rational number. Then $f - c$ is a partial function from X to $\mathbb{Q}$-PFuncs DOMS(Y).

Let us consider X, let Y be an integer-functions-membered set, let f be a partial function from X to Y, and let c be an integer number. Then $f - c$ is a partial function from X to $\mathbb{Z}$-PFuncs DOMS(Y).

We now state four propositions:

(49) If $f \neq \emptyset$ and f is non-empty and $f - c_1 = f - c_2$, then $c_1 = c_2$.

(50) $(f + c_1) - c_2 = f + (c_1 - c_2)$.

(51) $(f - c_1) + c_2 = f - (c_1 - c_2)$.

(52) $f - c_1 - c_2 = f - (c_1 + c_2)$.

Let us consider X, let Y be a complex-functions-membered set, let f be a partial function from X to Y, and let c be a complex number. The functor $f \cdot c$ yielding a function is defined as follows:

(Def. 39) $\mathrm{dom}(f \cdot c) = \mathrm{dom}\, f$ and for every set x such that $x \in \mathrm{dom}(f \cdot c)$ holds $(f \cdot c)(x) = c\, f(x)$.

Let us consider X, let Y be a complex-functions-membered set, let f be a partial function from X to Y, and let c be a complex number. Then $f \cdot c$ is a partial function from X to $\mathbb{C}$-PFuncs DOMS(Y).

Let us consider X, let Y be a real-functions-membered set, let f be a partial function from X to Y, and let c be a real number. Then $f \cdot c$ is a partial function from X to $\mathbb{R}$-PFuncs DOMS(Y).

Let us consider X, let Y be a rational-functions-membered set, let f be a partial function from X to Y, and let c be a rational number. Then $f \cdot c$ is a partial function from X to $\mathbb{Q}$-PFuncs DOMS(Y).

Let us consider X, let Y be an integer-functions-membered set, let f be a partial function from X to Y, and let c be an integer number. Then $f \cdot c$ is a partial function from X to $\mathbb{Z}$-PFuncs DOMS(Y).

Let us consider X, let Y be a natural-functions-membered set, let f be a partial function from X to Y, and let c be a natural number. Then $f \cdot c$ is a partial function from X to $\mathbb{N}$-PFuncs DOMS(Y).

The following two propositions are true:

(53) $f \cdot c_1 \cdot c_2 = f \cdot (c_1 \cdot c_2)$.

(54) If $f \neq \emptyset$ and f is non-empty and for every x such that $x \in \mathrm{dom}\, f$ holds $f(x)$ is non-empty and $f \cdot c_1 = f \cdot c_2$, then $c_1 = c_2$.

Let us consider X, let Y be a complex-functions-membered set, let f be a partial function from X to Y, and let c be a complex number. The functor f/c yielding a function is defined as follows:

(Def. 40) $f/c = f \cdot c^{-1}$.

One can prove the following propositions:

(55) $\mathrm{dom}(f/c) = \mathrm{dom}\, f$.

(56) If $x \in \mathrm{dom}(f/c)$, then $(f/c)(x) = c^{-1}\, f(x)$.

Let us consider X, let Y be a complex-functions-membered set, let f be a partial function from X to Y, and let c be a complex number. Then f/c is a partial function from X to $\mathbb{C}$-PFuncs DOMS(Y).

Let us consider X, let Y be a real-functions-membered set, let f be a partial function from X to Y, and let c be a real number. Then f/c is a partial function from X to $\mathbb{R}$-PFuncs DOMS(Y).

Let us consider X, let Y be a rational-functions-membered set, let f be a partial function from X to Y, and let c be a rational number. Then f/c is a partial function from X to $\mathbb{Q}$-PFuncs DOMS(Y).

The following propositions are true:

(57) $f/c_1/c_2 = f/(c_1 \cdot c_2)$.

(58) If $f \neq \emptyset$ and f is non-empty and for every x such that $x \in \operatorname{dom} f$ holds $f(x)$ is non-empty and $f/c_1 = f/c_2$, then $c_1 = c_2$.

Let us consider X, let Y be a complex-functions-membered set, let f be a partial function from X to Y, and let g be a complex-valued function. The functor $f + g$ yielding a function is defined as follows:

(Def. 41) $\operatorname{dom}(f+g) = \operatorname{dom} f \cap \operatorname{dom} g$ and for every set x such that $x \in \operatorname{dom}(f+g)$ holds $(f+g)(x) = f(x) + g(x)$.

Let us consider X, let Y be a complex-functions-membered set, let f be a partial function from X to Y, and let g be a complex-valued function. Then $f + g$ is a partial function from $X \cap \operatorname{dom} g$ to $\mathbb{C}$-PFuncs DOMS(Y).

Let us consider X, let Y be a real-functions-membered set, let f be a partial function from X to Y, and let g be a real-valued function. Then $f+g$ is a partial function from $X \cap \operatorname{dom} g$ to $\mathbb{R}$-PFuncs DOMS(Y).

Let us consider X, let Y be a rational-functions-membered set, let f be a partial function from X to Y, and let g be a rational-valued function. Then $f + g$ is a partial function from $X \cap \operatorname{dom} g$ to $\mathbb{Q}$-PFuncs DOMS(Y).

Let us consider X, let Y be an integer-functions-membered set, let f be a partial function from X to Y, and let g be an integer-valued function. Then $f + g$ is a partial function from $X \cap \operatorname{dom} g$ to $\mathbb{Z}$-PFuncs DOMS(Y).

Let us consider X, let Y be a natural-functions-membered set, let f be a partial function from X to Y, and let g be a natural-valued function. Then $f+g$ is a partial function from $X \cap \operatorname{dom} g$ to $\mathbb{N}$-PFuncs DOMS(Y).

Next we state two propositions:

(59) $f + g + h = f + (g + h)$.

(60) $-(f + g) = (-f) + -g$.

Let us consider X, let Y be a complex-functions-membered set, let f be a partial function from X to Y, and let g be a complex-valued function. The functor $f - g$ yields a function and is defined by:

(Def. 42) $f - g = f + -g$.

We now state two propositions:

(61) $\operatorname{dom}(f - g) = \operatorname{dom} f \cap \operatorname{dom} g$.

(62) If $x \in \operatorname{dom}(f - g)$, then $(f - g)(x) = f(x) - g(x)$.

Let us consider X, let Y be a complex-functions-membered set, let f be a partial function from X to Y, and let g be a complex-valued function. Then $f - g$ is a partial function from $X \cap \operatorname{dom} g$ to $\mathbb{C}$-PFuncs DOMS(Y).

Let us consider X, let Y be a real-functions-membered set, let f be a partial function from X to Y, and let g be a real-valued function. Then $f - g$ is a partial function from $X \cap \operatorname{dom} g$ to $\mathbb{R}$-PFuncs DOMS(Y).

Let us consider X, let Y be a rational-functions-membered set, let f be a partial function from X to Y, and let g be a rational-valued function. Then $f - g$ is a partial function from $X \cap \operatorname{dom} g$ to $\mathbb{Q}$-PFuncs DOMS(Y).

Let us consider X, let Y be an integer-functions-membered set, let f be a partial function from X to Y, and let g be an integer-valued function. Then $f - g$ is a partial function from $X \cap \operatorname{dom} g$ to $\mathbb{Z}$-PFuncs DOMS(Y).

The following propositions are true:

(63) $f - -g = f + g$.

(64) $-(f - g) = (-f) + g$.

(65) $(f + g) - h = f + (g - h)$.

(66) $(f - g) + h = f - (g - h)$.

(67) $f - g - h = f - (g + h)$.

Let us consider X, let Y be a complex-functions-membered set, let f be a partial function from X to Y, and let g be a complex-valued function. The functor $f \cdot g$ yielding a function is defined by:

(Def. 43) $\operatorname{dom}(f \cdot g) = \operatorname{dom} f \cap \operatorname{dom} g$ and for every set x such that $x \in \operatorname{dom}(f \cdot g)$ holds $(f \cdot g)(x) = f(x)\, g(x)$.

Let us consider X, let Y be a complex-functions-membered set, let f be a partial function from X to Y, and let g be a complex-valued function. Then $f \cdot g$ is a partial function from $X \cap \operatorname{dom} g$ to $\mathbb{C}$-PFuncs DOMS(Y).

Let us consider X, let Y be a real-functions-membered set, let f be a partial function from X to Y, and let g be a real-valued function. Then $f \cdot g$ is a partial function from $X \cap \operatorname{dom} g$ to $\mathbb{R}$-PFuncs DOMS(Y).

Let us consider X, let Y be a rational-functions-membered set, let f be a partial function from X to Y, and let g be a rational-valued function. Then $f \cdot g$ is a partial function from $X \cap \operatorname{dom} g$ to $\mathbb{Q}$-PFuncs DOMS(Y).

Let us consider X, let Y be an integer-functions-membered set, let f be a partial function from X to Y, and let g be an integer-valued function. Then $f \cdot g$ is a partial function from $X \cap \operatorname{dom} g$ to $\mathbb{Z}$-PFuncs DOMS(Y).

Let us consider X, let Y be a natural-functions-membered set, let f be a partial function from X to Y, and let g be a natural-valued function. Then $f \cdot g$ is a partial function from $X \cap \operatorname{dom} g$ to $\mathbb{N}$-PFuncs DOMS(Y).

Next we state three propositions:

(68) $f \cdot -g = (-f) \cdot g$.

(69) $f \cdot -g = -f \cdot g$.

(70) $f \cdot g \cdot h = f \cdot (g\, h)$.

Let us consider X, let Y be a complex-functions-membered set, let f be a partial function from X to Y, and let g be a complex-valued function. The functor f/g yields a function and is defined by:

(Def. 44) $f/g = f \cdot g^{-1}$.

Next we state two propositions:

(71) $\mathrm{dom}(f/g) = \mathrm{dom}\, f \cap \mathrm{dom}\, g$.

(72) If $x \in \mathrm{dom}(f/g)$, then $(f/g)(x) = f(x)/g(x)$.

Let us consider X, let Y be a complex-functions-membered set, let f be a partial function from X to Y, and let g be a complex-valued function. Then f/g is a partial function from $X \cap \mathrm{dom}\, g$ to $\mathbb{C}$-PFuncs DOMS(Y).

Let us consider X, let Y be a real-functions-membered set, let f be a partial function from X to Y, and let g be a real-valued function. Then f/g is a partial function from $X \cap \mathrm{dom}\, g$ to $\mathbb{R}$-PFuncs DOMS(Y).

Let us consider X, let Y be a rational-functions-membered set, let f be a partial function from X to Y, and let g be a rational-valued function. Then f/g is a partial function from $X \cap \mathrm{dom}\, g$ to $\mathbb{Q}$-PFuncs DOMS(Y).

Next we state the proposition

(73) $(f \cdot g)/h = f \cdot (g/h)$.

Let X_1, X_2 be sets, let Y_1, Y_2 be complex-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. The functor $f + g$ yielding a function is defined as follows:

(Def. 45) $\mathrm{dom}(f+g) = \mathrm{dom}\, f \cap \mathrm{dom}\, g$ and for every set x such that $x \in \mathrm{dom}(f+g)$ holds $(f+g)(x) = f(x) + g(x)$.

Let X_1, X_2 be sets, let Y_1, Y_2 be complex-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. Then $f + g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{C}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

Let X_1, X_2 be sets, let Y_1, Y_2 be real-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. Then $f + g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{R}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

Let X_1, X_2 be sets, let Y_1, Y_2 be rational-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. Then $f + g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{Q}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

Let X_1, X_2 be sets, let Y_1, Y_2 be integer-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to

Y_2. Then $f + g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{Z}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

Let X_1, X_2 be sets, let Y_1, Y_2 be natural-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. Then $f + g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{N}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

We now state three propositions:

(74) $f_1 + f_2 = f_2 + f_1$.

(75) $(f + f_1) + f_2 = f + (f_1 + f_2)$.

(76) $-(f_1 + f_2) = (-f_1) + -f_2$.

Let X_1, X_2 be sets, let Y_1, Y_2 be complex-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. The functor $f - g$ yields a function and is defined by:

(Def. 46) $\mathrm{dom}(f-g) = \mathrm{dom}\, f \cap \mathrm{dom}\, g$ and for every set x such that $x \in \mathrm{dom}(f-g)$ holds $(f - g)(x) = f(x) - g(x)$.

Let X_1, X_2 be sets, let Y_1, Y_2 be complex-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. Then $f - g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{C}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

Let X_1, X_2 be sets, let Y_1, Y_2 be real-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. Then $f - g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{R}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

Let X_1, X_2 be sets, let Y_1, Y_2 be rational-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. Then $f - g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{Q}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

Let X_1, X_2 be sets, let Y_1, Y_2 be integer-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. Then $f - g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{Z}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

One can prove the following propositions:

(77) $f_1 - f_2 = -(f_2 - f_1)$.

(78) $-(f_1 - f_2) = (-f_1) + f_2$.

(79) $(f + f_1) - f_2 = f + (f_1 - f_2)$.

(80) $(f - f_1) + f_2 = f - (f_1 - f_2)$.

(81) $f - f_1 - f_2 = f - (f_1 + f_2)$.

(82) $f - f_1 - f_2 = f - f_2 - f_1$.

Let X_1, X_2 be sets, let Y_1, Y_2 be complex-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2.

The functor $f \cdot g$ yields a function and is defined by:

(Def. 47) $\operatorname{dom}(f \cdot g) = \operatorname{dom} f \cap \operatorname{dom} g$ and for every set x such that $x \in \operatorname{dom}(f \cdot g)$ holds $(f \cdot g)(x) = f(x)\, g(x)$.

Let X_1, X_2 be sets, let Y_1, Y_2 be complex-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. Then $f \cdot g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{C}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

Let X_1, X_2 be sets, let Y_1, Y_2 be real-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. Then $f \cdot g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{R}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

Let X_1, X_2 be sets, let Y_1, Y_2 be rational-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. Then $f \cdot g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{Q}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

Let X_1, X_2 be sets, let Y_1, Y_2 be integer-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. Then $f \cdot g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{Z}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

Let X_1, X_2 be sets, let Y_1, Y_2 be natural-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. Then $f \cdot g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{N}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

We now state several propositions:

(83) $f_1 \cdot f_2 = f_2 \cdot f_1$.

(84) $(f \cdot f_1) \cdot f_2 = f \cdot (f_1 \cdot f_2)$.

(85) $(-f_1) \cdot f_2 = -f_1 \cdot f_2$.

(86) $f_1 \cdot -f_2 = -f_1 \cdot f_2$.

(87) $f \cdot (f_1 + f_2) = f \cdot f_1 + f \cdot f_2$.

(88) $(f_1 + f_2) \cdot f = f_1 \cdot f + f_2 \cdot f$.

(89) $f \cdot (f_1 - f_2) = f \cdot f_1 - f \cdot f_2$.

(90) $(f_1 - f_2) \cdot f = f_1 \cdot f - f_2 \cdot f$.

Let X_1, X_2 be sets, let Y_1, Y_2 be complex-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. The functor f/g yields a function and is defined by:

(Def. 48) $\operatorname{dom}(f/g) = \operatorname{dom} f \cap \operatorname{dom} g$ and for every set x such that $x \in \operatorname{dom}(f/g)$ holds $(f/g)(x) = f(x)/g(x)$.

Let X_1, X_2 be sets, let Y_1, Y_2 be complex-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2

to Y_2. Then f/g is a partial function from $X_1 \cap X_2$ to $\mathbb{C}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

Let X_1, X_2 be sets, let Y_1, Y_2 be real-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. Then f/g is a partial function from $X_1 \cap X_2$ to $\mathbb{R}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

Let X_1, X_2 be sets, let Y_1, Y_2 be rational-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. Then f/g is a partial function from $X_1 \cap X_2$ to $\mathbb{Q}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

One can prove the following propositions:

(91) $(-f_1)/f_2 = -f_1/f_2$.

(92) $f_1/-f_2 = -f_1/f_2$.

(93) $(f \cdot f_1)/f_2 = f \cdot (f_1/f_2)$.

(94) $(f/f_1) \cdot f_2 = (f \cdot f_2)/f_1$.

(95) $f/f_1/f_2 = f/(f_1 \cdot f_2)$.

(96) $(f_1 + f_2)/f = f_1/f + f_2/f$.

(97) $(f_1 - f_2)/f = f_1/f - f_2/f$.

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Received October 15, 2008