The Real Vector Spaces of Finite Sequences are Finite Dimensional

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Summary. In this paper we show the finite dimensionality of real linear spaces with their carriers equal \( \mathbb{R}^n \). We also give the standard basis of such spaces. For the set \( \mathbb{R}^n \) we introduce the concepts of linear manifold subsets and orthogonal subsets. The cardinality of orthonormal basis of discussed spaces is proved to equal \( n \).

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The articles [32], [7], [11], [33], [9], [2], [8], [5], [31], [4], [6], [18], [13], [22], [20], [14], [1], [21], [29], [28], [26], [3], [23], [10], [12], [30], [19], [34], [16], [17], [25], [15], [24], and [27] provide the notation and terminology for this paper.

1. Preliminaries

We use the following convention: \( i, j, n \) are elements of \( \mathbb{N} \), \( z \), \( B_0 \) are sets, and \( f, x_0 \) are real-valued finite sequences.

Next we state several propositions:

1. For all functions \( f, g \) holds \( \text{dom}(f \cdot g) = \text{dom } g \cap g^{-1}(\text{dom } f) \).

2. For every binary relation \( R \) and for every set \( Y \) such that \( \text{rng } R \subseteq Y \) holds \( R^{-1}(Y) = \text{dom } R \).
Let $X$ be a set, $Y$ be a non-empty set, and $f$ be a function from $X$ into $Y$. If $f$ is bijective, then $\overline{X} = \overline{Y}$.

For every element $x$ of $\mathbb{R}^0$ holds $x = \varepsilon_\mathbb{R}$.

For all elements $a, b, c$ of $\mathbb{R}^n$ holds $(a - b) + c + b = a + c$.

Let $f_1, f_2$ be finite sequences. One can verify that $\langle f_1, f_2 \rangle$ is finite sequence-like.

Let $D$ be a set and let $f_1, f_2$ be finite sequences of elements of $D$. Then $\langle f_1, f_2 \rangle$ is a finite sequence of elements of $D \times D$.

Let $h$ be a real-valued finite sequence. Let us observe that $h$ is increasing if and only if:

(Def. 1) For every $i$ such that $1 \leq i < \text{len } h$ holds $h(i) < h(i + 1)$.

One can prove the following four propositions:

(7) Let $h$ be a real-valued finite sequence. Suppose $h$ is increasing. Let given $i, j$. If $i < j$ and $1 \leq i$ and $j \leq \text{len } h$, then $h(i) < h(j)$.

(8) Let $h$ be a real-valued finite sequence. Suppose $h$ is increasing. Let given $i, j$. If $i \leq j$ and $1 \leq i$ and $j \leq \text{len } h$, then $h(i) \leq h(j)$.

(9) Let $h$ be a natural-valued finite sequence. Suppose $h$ is increasing. Let given $i$. If $1 \leq i \leq \text{len } h$ and $1 \leq h(1)$, then $i \leq h(i)$.

(10) Let $V$ be a real linear space and $X$ be a subspace of $V$. Suppose $V$ is strict and $X$ is strict and the carrier of $X$ = the carrier of $V$. Then $X = V$.

Let $D$ be a set, let $F$ be a finite sequence of elements of $D$, and let $h$ be a permutation of $\text{dom } F$. The functor $F \circ h$ yields a finite sequence of elements of $D$ and is defined as follows:

(Def. 2) $F \circ h = F \cdot h$.

One can prove the following propositions:

(11) Let $D$ be a non-empty set and $f$ be a finite sequence of elements of $D$. If $1 \leq i \leq \text{len } f$ and $1 \leq j \leq \text{len } f$, then $(\text{Swap}(f, i, j))(i) = f(j)$ and $(\text{Swap}(f, i, j))(j) = f(i)$.

(12) $\emptyset$ is a permutation of $\emptyset$.

(13) $\langle 1 \rangle$ is a permutation of $\{1\}$.

(14) For every finite sequence $h$ of elements of $\mathbb{R}$ holds $h$ is one-to-one iff $\text{sort}_a h$ is one-to-one.

(15) Let $h$ be a finite sequence of elements of $\mathbb{N}$. Suppose $h$ is one-to-one. Then there exists a permutation $h_3$ of $\text{dom } h$ and there exists a finite sequence $h_2$ of elements of $\mathbb{N}$ such that $h_2 = h \cdot h_3$ and $h_2$ is increasing and $\text{dom } h = \text{dom } h_2$ and $\text{rng } h = \text{rng } h_2$. 


2. Orthogonal Basis

Let $B_0$ be a set. We say that $B_0$ is $\mathbb{R}$-orthogonal if and only if:

(Def. 3) For all real-valued finite sequences $x, y$ such that $x, y \in B_0$ and $x \neq y$ holds $|\langle x, y \rangle| = 0$.

Let us observe that every set which is empty is also $\mathbb{R}$-orthogonal.

We now state the proposition

(16) $B_0$ is $\mathbb{R}$-orthogonal if and only if for all points $x, y$ of $E^n$ such that $x, y \in B_0$ and $x \neq y$ holds $x, y$ are orthogonal.

Let $B_0$ be a set. We say that $B_0$ is $\mathbb{R}$-normal if and only if:

(Def. 4) For every real-valued finite sequence $x$ such that $x \in B_0$ holds $|x| = 1$.

Let us observe that every set which is empty is also $\mathbb{R}$-normal.

Let us observe that there exists a set which is $\mathbb{R}$-normal.

Let $B_0, B_1$ be $\mathbb{R}$-normal sets. One can verify that $B_0 \cup B_1$ is $\mathbb{R}$-normal.

One can prove the following propositions:

(17) If $|f| = 1$, then $\{f\}$ is $\mathbb{R}$-normal.

(18) If $B_0$ is $\mathbb{R}$-normal and $|x_0| = 1$, then $B_0 \cup \{x_0\}$ is $\mathbb{R}$-normal.

Let $B_0$ be a set. We say that $B_0$ is $\mathbb{R}$-orthonormal if and only if:

(Def. 5) $B_0$ is $\mathbb{R}$-orthogonal and $\mathbb{R}$-normal.

Let us note that every set which is $\mathbb{R}$-orthonormal is also $\mathbb{R}$-orthogonal and $\mathbb{R}$-normal and every set which is $\mathbb{R}$-orthogonal and $\mathbb{R}$-normal is also $\mathbb{R}$-orthonormal.

Let us observe that $\{(1)\}$ is $\mathbb{R}$-orthonormal.

Let us observe that there exists a set which is $\mathbb{R}$-orthonormal and non empty.

Let us consider $n$. One can verify that there exists a subset of $\mathbb{R}^n$ which is $\mathbb{R}$-orthonormal.

Let us consider $n$ and let $B_0$ be a subset of $\mathbb{R}^n$. We say that $B_0$ is complete if and only if:

(Def. 6) For every $\mathbb{R}$-orthonormal subset $B$ of $\mathbb{R}^n$ such that $B_0 \subseteq B$ holds $B = B_0$.

Let $n$ be an element of $\mathbb{N}$ and let $B_0$ be a subset of $\mathbb{R}^n$. We say that $B_0$ is orthogonal basis if and only if:

(Def. 7) $B_0$ is $\mathbb{R}$-orthonormal and complete.

Let us consider $n$. One can verify that every subset of $\mathbb{R}^n$ which is orthogonal basis is also $\mathbb{R}$-orthonormal and complete and every subset of $\mathbb{R}^n$ which is $\mathbb{R}$-orthonormal and complete is also orthogonal basis.

The following propositions are true:

(19) For every subset $B_0$ of $\mathbb{R}^0$ such that $B_0$ is orthogonal basis holds $B_0 = \emptyset$. 

(20) Let $B_0$ be a subset of $\mathbb{R}^n$ and $y$ be an element of $\mathbb{R}^n$. Suppose $B_0$ is orthogonal basis and for every element $x$ of $\mathbb{R}^n$ such that $x \in B_0$ holds $|[x,y]| = 0$. Then $y = \langle 0, \ldots, 0 \rangle_n$.

3. Linear Manifolds

Let us consider $n$ and let $X$ be a subset of $\mathbb{R}^n$. We say that $X$ is linear manifold if and only if:

(Def. 8) For all elements $x, y$ of $\mathbb{R}^n$ and for all elements $a, b$ of $\mathbb{R}$ such that $x, y \in X$ holds $a \cdot x + b \cdot y \in X$.

Let us consider $n$. Observe that $\Omega_{\mathbb{R}^n}$ is linear manifold.

The following proposition is true

(21) $\{\langle 0, \ldots, 0 \rangle_n \}$ is linear manifold.

Let us consider $n$. Observe that $\{\langle 0, \ldots, 0 \rangle_n \}$ is linear manifold.

Let us consider $n$ and let $X$ be a subset of $\mathbb{R}^n$. The linear span of $X$ yielding a subset of $\mathbb{R}^n$ is defined by:

(Def. 9) The linear span of $X = \bigcap\{Y \subseteq \mathbb{R}^n: Y \text{ is linear manifold } \land X \subseteq Y\}$.

Let us consider $n$ and let $X$ be a subset of $\mathbb{R}^n$. Observe that the linear span of $X$ is linear manifold.

Let us consider $n$ and let $f$ be a finite sequence of elements of $\mathbb{R}^n$. The functor $\sum f$ yielding an element of $\mathbb{R}^n$ is defined as follows:

(Def. 10) (i) There exists a finite sequence $g$ of elements of $\mathbb{R}^n$ such that $\text{len } f = \text{len } g$ and $f(1) = g(1)$ and for every natural number $i$ such that $1 \leq i < \text{len } f$ holds $g(i + 1) = g_i + f_{i+1}$ and $\sum f = g(\text{len } f)$ if $\text{len } f > 0$, (ii) $\sum f = \langle 0, \ldots, 0 \rangle_n$, otherwise.

Let $n$ be a natural number and let $f$ be a finite sequence of elements of $\mathbb{R}^n$. The functor $\text{accum } f$ yields a finite sequence of elements of $\mathbb{R}^n$ and is defined as follows:

(Def. 11) $\text{len } f = \text{len } \text{accum } f$ and $f(1) = (\text{accum } f)(1)$ and for every natural number $i$ such that $1 \leq i < \text{len } f$ holds $(\text{accum } f)(i + 1) = (\text{accum } f)_i + f_{i+1}$.

We now state several propositions:

(22) For every finite sequence $f$ of elements of $\mathbb{R}^n$ such that $\text{len } f > 0$ holds $(\text{accum } f)(\text{len } f) = \sum f$.

(23) For all finite sequences $F, F_2$ of elements of $\mathbb{R}^n$ and for every permutation $h$ of $\text{dom } F$ such that $F_2 = F \circ h$ holds $\sum F_2 = \sum F$.

(24) For every element $k$ of $\mathbb{N}$ holds $\sum k \mapsto \langle 0, \ldots, 0 \rangle_n = \langle 0, \ldots, 0 \rangle_n$.
(25) Let \( g \) be a finite sequence of elements of \( \mathbb{R}^n \), \( h \) be a finite sequence of elements of \( \mathbb{N} \), and \( F \) be a finite sequence of elements of \( \mathbb{R}^n \). Suppose \( h \) is increasing and \( \text{rng} \ h \subseteq \text{dom} \ g \) and for every element \( i \) of \( \mathbb{N} \) such that \( i \in \text{dom} \ g \) and \( i \notin \text{rng} \ h \) holds \( g(i) = (0, \ldots, 0) \). Then

\[
\sum g = \sum F.
\]

(26) Let \( g \) be a finite sequence of elements of \( \mathbb{R}^n \), \( h \) be a finite sequence of elements of \( \mathbb{N} \), and \( F \) be a finite sequence of elements of \( \mathbb{R}^n \). Suppose \( h \) is one-to-one and \( \text{rng} \ h \subseteq \text{dom} \ g \) and \( F = g \cdot h \) and for every element \( i \) of \( \mathbb{N} \) such that \( i \in \text{dom} \ g \) and \( i \notin \text{rng} \ h \) holds \( g(i) = (0, \ldots, 0) \). Then

\[
\sum g = \sum F.
\]

4. Standard Basis

Let us consider \( n, i \). Then the base finite sequence of \( n \) and \( i \) is an element of \( \mathbb{R}^n \).

The following propositions are true:

(27) Let \( i_1, i_2 \) be elements of \( \mathbb{N} \). Suppose that
   (i) \( 1 \leq i_1 \),
   (ii) \( i_1 \leq n \),
   (iii) \( 1 \leq i_2 \),
   (iv) \( i_2 \leq n \), and
   (v) the base finite sequence of \( n \) and \( i_1 \) = the base finite sequence of \( n \) and \( i_2 \).

Then \( i_1 = i_2 \).

(28) \( \sum (\text{the base finite sequence of } n \text{ and } i) = \sum (\text{the base finite sequence of } n \text{ and } i) \).

(29) \( \sum \text{the base finite sequence of } n \text{ and } i = 1 \).

(30) \( |\text{the base finite sequence of } n \text{ and } i| = 1 \).

(31) Suppose \( 1 \leq i \leq n \) and \( 1 \leq j \leq n \) and \( i \neq j \). Then \( |(\text{the base finite sequence of } n \text{ and } i, \text{the base finite sequence of } n \text{ and } j)| = 0 \).

(32) For every element \( x \) of \( \mathbb{R}^n \) such that \( 1 \leq i \leq n \) holds \( |(x, \text{the base finite sequence of } n \text{ and } i)| = x(i) \).

Let us consider \( n \) and let \( x_0 \) be an element of \( \mathbb{R}^n \). The functor \( \text{ProjFinSeq} \ x_0 \) yields a finite sequence of elements of \( \mathbb{R}^n \) and is defined by the conditions (Def. 12).

(Def. 12)(i) \( \text{len} \ \text{ProjFinSeq} \ x_0 = n \), and
   (ii) for every \( i \) such that \( 1 \leq i \leq n \) holds \( (\text{ProjFinSeq} \ x_0)(i) = |(x_0, \text{the base finite sequence of } n \text{ and } i)| \cdot \text{the base finite sequence of } n \text{ and } i \).
The following proposition is true

(33) For every element \( x_0 \) of \( \mathcal{R}^n \) holds \( x_0 = \sum \text{ProjFinSeq} x_0 \).

Let us consider \( n \). The functor \( \mathbb{R}\text{N-Base} \) \( n \) yields a subset of \( \mathcal{R}^n \) and is defined by:

(Def. 13) \( \mathbb{R}\text{N-Base} \) \( n \) = \{the base finite sequence of \( n \) and \( i \); \( i \) ranges over elements of \( \mathbb{N} \); \( 1 \leq i \land i \leq n \} \).

Next we state the proposition

(34) For every non zero element \( n \) of \( \mathbb{N} \) holds \( \mathbb{R}\text{N-Base} \) \( n \) \( \neq \emptyset \).

Let us mention that \( \mathbb{R}\text{N-Base} \) \( 0 \) is empty.

Let \( n \) be a non zero element of \( \mathbb{N} \). Note that \( \mathbb{R}\text{N-Base} \) \( n \) is non empty.

Let us consider \( n \). Observe that \( \mathbb{R}\text{N-Base} \) \( n \) is orthogonal basis.

Let us consider \( n \). Observe that there exists a subset of \( \mathcal{R}^n \) which is orthogonal basis.

Let us consider \( n \). An orthogonal basis of \( n \) is an orthogonal basis subset of \( \mathcal{R}^n \).

Let \( n \) be a non zero element of \( \mathbb{N} \). Observe that every orthogonal basis of \( n \) is non empty.

5. Finite Real Unitary Spaces and Finite Real Linear Spaces

Let \( n \) be an element of \( \mathbb{N} \). Observe that \( \langle \mathcal{E}^n, (\cdot, \cdot) \rangle \) is constituted finite sequences. Let \( n \) be an element of \( \mathbb{N} \). One can check that every element of \( \langle \mathcal{E}^n, (\cdot, \cdot) \rangle \) is real-valued.

Let \( n \) be an element of \( \mathbb{N} \), let \( x, y \) be vectors of \( \langle \mathcal{E}^n, (\cdot, \cdot) \rangle \), and let \( a, b \) be real-valued functions. One can verify that \( x + y \) and \( a + b \) can be identified when \( x = a \) and \( y = b \).

Let \( n \) be an element of \( \mathbb{N} \), let \( x \) be a vector of \( \langle \mathcal{E}^n, (\cdot, \cdot) \rangle \), let \( y \) be a real-valued function, and let \( a, b \) be elements of \( \mathbb{R} \). Observe that \( a \cdot x \) and \( b \cdot y \) can be identified when \( a = b \) and \( x = y \).

Let \( n \) be an element of \( \mathbb{N} \), let \( x \) be a vector of \( \langle \mathcal{E}^n, (\cdot, \cdot) \rangle \), and let \( a \) be a real-valued function. Observe that \( -x \) and \( -a \) can be identified when \( x = a \).

Let \( n \) be an element of \( \mathbb{N} \), let \( x, y \) be vectors of \( \langle \mathcal{E}^n, (\cdot, \cdot) \rangle \), and let \( a, b \) be real-valued functions. One can check that \( x - y \) and \( a - b \) can be identified when \( x = a \) and \( y = b \). The following three propositions are true:

(35) Let \( n \) be an element of \( \mathbb{N} \), \( x, y \) be elements of \( \mathcal{R}^n \), and \( u, v \) be points of \( \langle \mathcal{E}^n, (\cdot, \cdot) \rangle \). If \( x = u \) and \( y = v \), then \( \otimes_{\mathcal{E}^n} (\{u, v\}) = |(x, y)| \).

(36) Let \( n, j \) be elements of \( \mathbb{N} \), \( F \) be a finite sequence of elements of the carrier of \( \langle \mathcal{E}^n, (\cdot, \cdot) \rangle \), \( B_2 \) be a subset of \( \langle \mathcal{E}^n, (\cdot, \cdot) \rangle \), \( v_0 \) be an element of \( \langle \mathcal{E}^n, (\cdot, \cdot) \rangle \), and \( l \) be a linear combination of \( B_2 \). Suppose \( F \) is one-to-one and \( B_2 \) is \( \mathbb{R} \)-orthogonal and \( \text{rng} \, F = \text{the support of} \, l \) and \( v_0 \in B_2 \) and \( j \in \text{dom}(l \, F) \) and \( v_0 = F(j) \). Then \( \otimes_{\mathcal{E}^n} (\{v_0, \sum l \, F\}) = \otimes_{\mathcal{E}^n} (\{v_0, l(F_j \cdot v_0)\}) \).
is also linearly independent.

Let $A$ be a set. Note that $\mathbb{R}^A$ is constituted functions.

Let us consider $n$. Observe that $\mathbb{R}^{\text{Seg}^n}$ is constituted finite sequences.

Let $A$ be a set. One can verify that every element of $\mathbb{R}^A$ is real-valued.

Let $A$ be a set, let $x$, $y$ be vectors of $\mathbb{R}^A$, and let $a$, $b$ be real-valued functions.

Observe that $x + y$ and $a + b$ can be identified when $x = a$ and $y = b$.

Let $A$ be a set, let $x$ be a vector of $\mathbb{R}^A$, let $y$ be a real-valued function, and let $a$, $b$ be elements of $\mathbb{R}$. Observe that $a \cdot x$ and $b y$ can be identified when $a = b$ and $x = y$.

Let $A$ be a set, let $x$ be a vector of $\mathbb{R}^A$, and let $a$ be a real-valued function.

One can check that $-x$ and $-a$ can be identified when $x = a$.

Let $A$ be a set, let $x$, $y$ be vectors of $\mathbb{R}^A$, and let $a$, $b$ be real-valued functions.

Observe that $x - y$ and $a - b$ can be identified when $x = a$ and $y = b$.

The following propositions are true:

(38) Let $X$ be a subspace of $\mathbb{R}^{\text{Seg}^n}$, $x$ be an element of $\mathbb{R}^n$, and $a$ be a real number. If $x \in$ the carrier of $X$, then $a \cdot x \in$ the carrier of $X$.

(39) Let $X$ be a subspace of $\mathbb{R}^{\text{Seg}^n}$ and $x$, $y$ be elements of $\mathbb{R}^n$. Suppose $x \in$ the carrier of $X$ and $y \in$ the carrier of $X$. Then $x + y \in$ the carrier of $X$.

(40) Let $X$ be a subspace of $\mathbb{R}^{\text{Seg}^n}$, $x$, $y$ be elements of $\mathbb{R}^n$, and $a$, $b$ be real numbers. Suppose $x \in$ the carrier of $X$ and $y \in$ the carrier of $X$. Then $a \cdot x + b \cdot y \in$ the carrier of $X$.

(41) For all elements $x$, $y$ of $\mathbb{R}^n$ and for all points $u$, $v$ of $\mathbb{R}^{\text{Seg}^n}$ such that $x = u$ and $y = v$ holds $\otimes \epsilon^n((u, v)) = |(x, y)|$.

(42) Let $F$ be a finite sequence of elements of the carrier of $\mathbb{R}^{\text{Seg}^n}$, $B_2$ be a subset of $\mathbb{R}^{\text{Seg}^n}$, $v_0$ be an element of $\mathbb{R}^{\text{Seg}^n}$, and $l$ be a linear combination of $B_2$. Suppose $F$ is one-to-one and $B_2$ is $\mathbb{R}$-orthogonal and $\text{rng } F = \text{the support of } l \text{ and } v_0 \in B_2 \text{ and } j \in \text{dom}(l F)$ and $v_0 = F(j)$. Then $\otimes \epsilon^n((v_0, \sum l F)) = \otimes \epsilon^n((v_0, l(F_j) \cdot v_0))$.

Let us consider $n$. Note that every subset of $\mathbb{R}^{\text{Seg}^n}$ which is $\mathbb{R}$-orthonormal is also linearly independent.

Let $n$ be an element of $\mathbb{N}$. Note that every subset of $\langle \mathcal{E}^n, \langle \cdot \rangle \rangle$ which is $\mathbb{R}$-orthonormal is also linearly independent. Next we state the proposition

(43) Let $B_2$ be a subset of $\mathbb{R}^{\text{Seg}^n}$, $x$, $y$ be elements of $\mathbb{R}^n$, and $a$ be a real number. If $B_2$ is linearly independent and $x$, $y \in B_2$ and $y = a \cdot x$, then $x = y$. 
6. Finite Dimensionality of the Spaces

Let us consider $n$. One can check that $\mathbb{R}^n$ is finite.

The following propositions are true:

(44) $\text{card } \mathbb{R}^n = n$.

(45) Let $f$ be a finite sequence of elements of $\mathbb{R}^n$ and $g$ be a finite sequence of elements of the carrier of $\mathbb{R}^n$. If $f = g$, then $\sum f = \sum g$.

(46) Let $x_0$ be an element of $\mathbb{R}^n$ and $B$ be a subset of $\mathbb{R}^n$. If $B = \mathbb{R}^n$, then there exists a linear combination $l$ of $B$ such that $x_0 = \sum l$.

(47) Let $n$ be an element of $\mathbb{N}$, $x_0$ be an element of $\langle E^n, (\cdot | \cdot) \rangle$, and $B$ be a subset of $\langle E^n, (\cdot | \cdot) \rangle$. If $B = \mathbb{R}^n$, then there exists a linear combination $l$ of $B$ such that $x_0 = \sum l$.

(48) For every subset $B$ of $\mathbb{R}^n$ such that $B = \mathbb{R}^n$ holds $B$ is a basis of $\mathbb{R}^n$.

Let us consider $n$. Observe that $\mathbb{R}^n$ is finite dimensional.

We now state several propositions:

(49) $\text{dim}(\mathbb{R}^n) = n$.

(50) For every subset $B$ of $\mathbb{R}^n$ such that $B$ is a basis of $\mathbb{R}^n$ holds $B = \mathbb{R}^n$.

(51) $\emptyset$ is a basis of $\mathbb{R}^0$.

(52) For every element $n$ of $\mathbb{N}$ holds $\mathbb{R}^n$ is a basis of $\langle E^n, (\cdot | \cdot) \rangle$.

(53) Every orthogonal basis of $n$ is a basis of $\mathbb{R}^n$.

Let $n$ be an element of $\mathbb{N}$. Note that $\langle E^n, (\cdot | \cdot) \rangle$ is finite dimensional.

We now state two propositions:

(54) For every element $n$ of $\mathbb{N}$ holds $\text{dim}(\langle E^n, (\cdot | \cdot) \rangle) = n$.

(55) For every orthogonal basis $B$ of $n$ holds $\overline{B} = n$.

References


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