FORMALIZED MATHEMATICS
Volume 13, Number 4, Pages 511–513
University of Białystok, 2005

Formulas and Identities of Hyperbolic Functions

Pacharapokin Chanapat
Shinshu University
Nagano, Japan

Hiroshi Yamazaki
Shinshu University
Nagano, Japan

Summary. In this article, we proved formulas of hyperbolic sine, hyperbolic cosine and hyperbolic tangent, and their identities.

MML identifier: SIN_COS8, version: 7.6.01 4.46.926

The papers [1], [3], [6], [5], [7], [4], and [2] provide the terminology and notation for this paper.

We follow the rules: x, y, z, w are real numbers and n is a natural number.

One can prove the following propositions:

(1) $\tanh x = \frac{\sinh x}{\cosh x}$ and $\tanh 0 = 0$.

(2) $\sinh x = \frac{1}{\operatorname{cosech} x}$ and $\cosh x = \frac{1}{\operatorname{sech} x}$ and $\tanh x = \frac{1}{\coth x}$.

(3) $\operatorname{sech} x \leq 1$ and $0 < \operatorname{sech} x$ and $\operatorname{sech} 0 = 1$.

(4) If $x \geq 0$, then $\tanh x \geq 0$.

(5) $\cosh x = \frac{1}{\sqrt{1-(\tanh x)^2}}$ and $\sinh x = \frac{\tanh x}{\sqrt{1-(\tanh x)^2}}$.

(6) $(\cosh x + \sinh x)^n = \cosh(n \cdot x) + \sinh(n \cdot x)$ and $(\cosh x - \sinh x)^n = \cosh(n \cdot x) - \sinh(n \cdot x)$.

(7)(i) $\exp x = \cosh x + \sinh x$,

(ii) $\exp(-x) = \cosh x - \sinh x$,

(iii) $\exp x = \frac{\cosh(\frac{x}{2})+\sinh(\frac{x}{2})}{\cosh(\frac{x}{2})-\sinh(\frac{x}{2})}$,

(iv) $\exp(-x) = \frac{\cosh(\frac{x}{2})-\sinh(\frac{x}{2})}{\cosh(\frac{x}{2})+\sinh(\frac{x}{2})}$,

(v) $\exp x = \frac{1+\tanh(\frac{x}{2})}{1-\tanh(\frac{x}{2})}$, and

(vi) $\exp(-x) = \frac{1-\tanh(\frac{x}{2})}{1+\tanh(\frac{x}{2})}$.

© 2005 University of Białystok
ISSN 1426–2630

(8) If $x \neq 0$, then $\exp x = \frac{\coth(\frac{x}{2})+1}{\coth(\frac{x}{2})-1}$ and $\exp(-x) = \frac{\coth(\frac{x}{2})-1}{\coth(\frac{x}{2})+1}$.

(9) $\frac{\cosh x+\sinh x}{\cosh x-\sinh x} = \frac{1+\tanh x}{1-\tanh x}$.

(10) If $y \neq 0$, then $\coth y + \tanh z = \frac{\cosh(y+z)}{\sinh y \cdot \cosh z}$ and $\coth y - \tanh z = \frac{\cosh(y-z)}{\sinh y \cdot \cosh z}$.

(11) $\sinh y \cdot \sinh z = \frac{1}{2} \cdot (\cosh(y+z) - \cosh(y-z))$ and $\sinh y \cdot \cosh z = \frac{1}{2} \cdot (\sinh(y+z) + \sinh(y-z))$ and $\cosh y \cdot \sinh z = \frac{1}{2} \cdot (\sinh(y+z) - \sinh(y-z))$ and $\cosh y \cdot \cosh z = \frac{1}{2} \cdot (\cosh(y+z) + \cosh(y-z))$.

(12) $(\sinh y)^{\mathbf{2}} - (\cosh z)^{\mathbf{2}} = \sinh(y+z) \cdot \sinh(y-z) - 1$.

(13) $(\sinh y - \sinh z)^{\mathbf{2}} - (\cosh y - \cosh z)^{\mathbf{2}} = 4 \cdot (\sinh(\frac{y-z}{2}))^{\mathbf{2}}$ and $(\cosh y + \cosh z)^{\mathbf{2}} - (\sinh y + \sinh z)^{\mathbf{2}} = 4 \cdot (\cosh(\frac{y-z}{2}))^{\mathbf{2}}$.

(14) $\frac{\sinh y+\sinh z}{\sinh y-\sinh z} = \tanh(\frac{y+z}{2}) \cdot \coth(\frac{y-z}{2})$.

(15) $\frac{\cosh y+\cosh z}{\cosh y-\cosh z} = \coth(\frac{y+z}{2}) \cdot \coth(\frac{y-z}{2})$.

(16) If $y - z \neq 0$, then $\frac{\sinh y+\sinh z}{\cosh y+\cosh z} = \frac{\cosh y-\cosh z}{\sinh y-\sinh z}$.

(17) If $y + z \neq 0$, then $\frac{\sinh y-\sinh z}{\cosh y+\cosh z} = \frac{\cosh y-\cosh z}{\sinh y+\sinh z}$.

(18) $\frac{\sinh y+\sinh z}{\cosh y+\cosh z} = \tanh(\frac{y}{2} + \frac{z}{2})$ and $\frac{\sinh y-\sinh z}{\cosh y+\cosh z} = \tanh(\frac{y}{2} - \frac{z}{2})$.

(19) $\frac{\tanh y+\tanh z}{\tanh y-\tanh z} = \frac{\sinh(y+z)}{\sinh(y-z)}$.

(20) $\frac{\sinh(y-z)+\sinh y+\sinh(y+z)}{\cosh(y-z)+\cosh y+\cosh(y+z)} = \tanh y$.

(21)(i) $\sinh(y+z+w) = (\tanh y + \tanh z + \tanh w + \tanh y \cdot \tanh z \cdot \tanh w) \cdot \cosh y \cdot \cosh z \cdot \cosh w$,

(ii) $\cosh(y+z+w) = (1 + \tanh y \cdot \tanh z + \tanh z \cdot \tanh w + \tanh w \cdot \tanh y) \cdot \cosh y \cdot \cosh z \cdot \cosh w$, and

(iii) $\tanh(y+z+w) = \frac{\tanh y+\tanh z+\tanh w+\tanh y \cdot \tanh z \cdot \tanh w}{1+\tanh z \cdot \tanh w+\tanh w \cdot \tanh y+\tanh y \cdot \tanh z}$.

(22) $\cosh(2 \cdot y) + \cosh(2 \cdot z) + \cosh(2 \cdot w) + \cosh(2 \cdot (y+z+w)) = 4 \cdot \cosh(z+w) \cdot \cosh(w+y) \cdot \cosh(y+z)$.

(23) $\sinh y \cdot \sinh z \cdot \sinh(z-y) + \sinh z \cdot \sinh w \cdot \sinh(w-z) + \sinh w \cdot \sinh y \cdot \sinh(y-w) + \sinh(z-y) \cdot \sinh(w-z) \cdot \sinh(y-w) = 0$.

(24) If $x \geq 0$, then $\sinh(\frac{x}{2}) = \sqrt{\frac{\cosh x-1}{2}}$.

(25) If $x < 0$, then $\sinh(\frac{x}{2}) = -\sqrt{\frac{\cosh x-1}{2}}$.

(26) $\sinh(2 \cdot x) = 2 \cdot \sinh x \cdot \cosh x$ and $\cosh(2 \cdot x) = 2 \cdot (\cosh x)^{\mathbf{2}} - 1$ and $\tanh(2 \cdot x) = \frac{2 \cdot \tanh x}{1+(\tanh x)^{\mathbf{2}}}$.

(27) $\sinh(2 \cdot x) = \frac{2 \cdot \tanh x}{1-(\tanh x)^{\mathbf{2}}}$ and $\sinh(3 \cdot x) = \sinh x \cdot (4 \cdot (\cosh x)^{\mathbf{2}} - 1)$ and $\sinh(3 \cdot x) = 3 \cdot \sinh x - 2 \cdot \sinh x \cdot (1 - \cosh(2 \cdot x))$ and $\cosh(2 \cdot x) = 1 + 2 \cdot (\sinh x)^{\mathbf{2}}$ and $\cosh(2 \cdot x) = (\cosh x)^{\mathbf{2}} + (\sinh x)^{\mathbf{2}}$ and $\cosh(2 \cdot x) = \frac{1+(\tanh x)^{\mathbf{2}}}{1-(\tanh x)^{\mathbf{2}}}$ and $\cosh(3 \cdot x) = \cosh x \cdot (4 \cdot (\sinh x)^{\mathbf{2}} + 1)$ and $\tanh(3 \cdot x) = \frac{3 \cdot \tanh x + (\tanh x)^3}{1+3 \cdot (\tanh x)^{\mathbf{2}}}$.

(28) $\frac{\sinh(5\cdot x)+2\cdot\sinh(3\cdot x)+\sinh x}{\sinh(7\cdot x)+2\cdot\sinh(5\cdot x)+\sinh(3\cdot x)} = \frac{\sinh(3\cdot x)}{\sinh(5\cdot x)}$.

(29) If $x \geq 0$, then $\tanh(\frac{x}{2}) = \sqrt{\frac{\cosh x-1}{\cosh x+1}}$.

(30) If $x < 0$, then $\tanh(\frac{x}{2}) = -\sqrt{\frac{\cosh x-1}{\cosh x+1}}$.

(31)(i) $(\sinh x)^3 = \frac{\sinh(3\cdot x)-3\cdot\sinh x}{4}$,

(ii) $(\sinh x)^4 = \frac{(\cosh(4\cdot x)-4\cdot\cosh(2\cdot x))+3}{8}$,

(iii) $(\sinh x)^5 = \frac{(\sinh(5\cdot x)-5\cdot\sinh(3\cdot x))+10\cdot\sinh x}{16}$,

(iv) $(\sinh x)^6 = \frac{((\cosh(6\cdot x)-6\cdot\cosh(4\cdot x))+15\cdot\cosh(2\cdot x))-10}{32}$,

(v) $(\sinh x)^7 = \frac{((\sinh(7\cdot x)-7\cdot\sinh(5\cdot x))+21\cdot\sinh(3\cdot x))-35\cdot\sinh x}{64}$, and

(vi) $(\sinh x)^8 = \frac{(((\cosh(8\cdot x)-8\cdot\cosh(6\cdot x))+28\cdot\cosh(4\cdot x))-56\cdot\cosh(2\cdot x))+35}{128}$.

(32)(i) $(\cosh x)^3 = \frac{\cosh(3\cdot x)+3\cdot\cosh x}{4}$,

(ii) $(\cosh x)^4 = \frac{\cosh(4\cdot x)+4\cdot\cosh(2\cdot x)+3}{8}$,

(iii) $(\cosh x)^5 = \frac{\cosh(5\cdot x)+5\cdot\cosh(3\cdot x)+10\cdot\cosh x}{16}$,

(iv) $(\cosh x)^6 = \frac{\cosh(6\cdot x)+6\cdot\cosh(4\cdot x)+15\cdot\cosh(2\cdot x)+10}{32}$,

(v) $(\cosh x)^7 = \frac{\cosh(7\cdot x)+7\cdot\cosh(5\cdot x)+21\cdot\cosh(3\cdot x)+35\cdot\cosh x}{64}$, and

(vi) $(\cosh x)^8 = \frac{\cosh(8\cdot x)+8\cdot\cosh(6\cdot x)+28\cdot\cosh(4\cdot x)+56\cdot\cosh(2\cdot x)+35}{128}$.

(33) $\cosh(2\cdot y)+\cos(2\cdot z) = 2+2\cdot((\sinh y)^{\mathbf{2}}-(\sin z)^{\mathbf{2}})$ and $\cosh(2\cdot y)-\cos(2\cdot z) = 2\cdot((\sinh y)^{\mathbf{2}}+(\sin z)^{\mathbf{2}})$.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[2] Yuzhong Ding and Xiquan Liang. Formulas and identities of trigonometric functions. *Formalized Mathematics*, 12(**3**):243–246, 2004.

[3] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[4] Takashi Mitsuishi and Yuguang Yang. Properties of the trigonometric function. *Formalized Mathematics*, 8(**1**):103–106, 1999.

[5] Konrad Raczkowski. Integer and rational exponents. *Formalized Mathematics*, 2(**1**):125–130, 1991.

[6] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.

[7] Yuguang Yang and Yasunari Shidama. Trigonometric functions and existence of circle ratio. *Formalized Mathematics*, 7(**2**):255–263, 1998.

Received November 7, 2005