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Some Differentiable Formulas of Special Functions

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Summary. This article contains some differentiable formulas of special functions.

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The terminology and notation used in this paper are introduced in the following papers: [13], [15], [16], [2], [4], [10], [12], [3], [1], [6], [9], [7], [8], [11], [17], [5], and [14].

For simplicity, we use the following convention: x, a, b are real numbers, n is a natural number, Z is an open subset of $\mathbb{R}$, and f, f_1, f_2, g are partial functions from $\mathbb{R}$ to $\mathbb{R}$.

Next we state a number of propositions:

(1) Suppose $Z \subseteq \operatorname{dom}(\frac{f_1}{f_2})$ and for every x such that $x \in Z$ holds $f_1(x) = a + x$ and $f_2(x) = a - x$ and $f_2(x) \neq 0$. Then $\frac{f_1}{f_2}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{f_1}{f_2})'_{\restriction Z}(x) = \frac{2 \cdot a}{(a-x)^2}$.

(2) Suppose $Z \subseteq \operatorname{dom}(\frac{f_1}{f_2})$ and for every x such that $x \in Z$ holds $f_1(x) = x - a$ and $f_2(x) = x + a$ and $f_2(x) \neq 0$. Then $\frac{f_1}{f_2}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{f_1}{f_2})'_{\restriction Z}(x) = \frac{2 \cdot a}{(x+a)^2}$.

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(3) Suppose $Z \subseteq \operatorname{dom}(\frac{f_1}{f_2})$ and for every x such that $x \in Z$ holds $f_1(x) = x - a$ and $f_2(x) = x - b$ and $f_2(x) \neq 0$. Then $\frac{f_1}{f_2}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{f_1}{f_2})'_{\restriction Z}(x) = \frac{a-b}{(x-b)^{\mathbf{2}}}$.

(4) Suppose $Z \subseteq \operatorname{dom} f$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then $\frac{1}{f}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{1}{f})'_{\restriction Z}(x) = -\frac{1}{x^{\mathbf{2}}}$.

(5) Suppose $Z \subseteq \operatorname{dom}((\text{the function sin}) \cdot \frac{1}{f})$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) (the function sin) $\cdot \frac{1}{f}$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function sin}) \cdot \frac{1}{f})'_{\restriction Z}(x) = -\frac{1}{x^{\mathbf{2}}} \cdot (\text{the function cos})(\frac{1}{x})$.

(6) Suppose $Z \subseteq \operatorname{dom}((\text{the function cos}) \cdot \frac{1}{f})$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) (the function cos) $\cdot \frac{1}{f}$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function cos}) \cdot \frac{1}{f})'_{\restriction Z}(x) = \frac{1}{x^{\mathbf{2}}} \cdot (\text{the function sin})(\frac{1}{x})$.

(7) Suppose $Z \subseteq \operatorname{dom}(\operatorname{id}_Z((\text{the function sin}) \cdot \frac{1}{f}))$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) $\operatorname{id}_Z((\text{the function sin}) \cdot \frac{1}{f})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\operatorname{id}_Z((\text{the function sin}) \cdot \frac{1}{f}))'_{\restriction Z}(x) = (\text{the function sin})(\frac{1}{x}) - \frac{1}{x} \cdot (\text{the function cos})(\frac{1}{x})$.

(8) Suppose $Z \subseteq \operatorname{dom}(\operatorname{id}_Z((\text{the function cos}) \cdot \frac{1}{f}))$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) $\operatorname{id}_Z((\text{the function cos}) \cdot \frac{1}{f})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\operatorname{id}_Z((\text{the function cos}) \cdot \frac{1}{f}))'_{\restriction Z}(x) = (\text{the function cos})(\frac{1}{x}) + \frac{1}{x} \cdot (\text{the function sin})(\frac{1}{x})$.

(9) Suppose $Z \subseteq \operatorname{dom}(((\text{the function sin}) \cdot \frac{1}{f})((\text{the function cos}) \cdot \frac{1}{f}))$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) $((\text{the function sin}) \cdot \frac{1}{f})((\text{the function cos}) \cdot \frac{1}{f})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(((\text{the function sin}) \cdot \frac{1}{f})((\text{the function cos}) \cdot \frac{1}{f}))'_{\restriction Z}(x) = \frac{1}{x^{\mathbf{2}}} \cdot ((\text{the function sin})(\frac{1}{x})^{\mathbf{2}} - (\text{the function cos})(\frac{1}{x})^{\mathbf{2}})$.

(10) Suppose $Z \subseteq \operatorname{dom}(((\text{the function sin}) \cdot f)((\mathbb{Z}^n) \cdot (\text{the function sin})))$ and $n \geq 1$ and for every x such that $x \in Z$ holds $f(x) = n \cdot x$. Then

(i) $((\text{the function sin}) \cdot f)((\mathbb{Z}^n) \cdot (\text{the function sin}))$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(((\text{the function sin}) \cdot f)((\mathbb{Z}^n) \cdot (\text{the function sin})))'_{\restriction Z}(x) = n \cdot (\text{the function sin})(x)^{n-1}_{\mathbb{Z}} \cdot (\text{the function sin})((n+1) \cdot x)$.

(11) Suppose $Z \subseteq \mathrm{dom}(((\text{the function cos}) \cdot f)\,((\binom{n}{\mathbb{Z}}) \cdot (\text{the function sin})))$ and $n \geq 1$ and for every x such that $x \in Z$ holds $f(x) = n \cdot x$. Then

(i) $((\text{the function cos}) \cdot f)\,((\binom{n}{\mathbb{Z}}) \cdot (\text{the function sin}))$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(((\text{the function cos}) \cdot f)\,((\binom{n}{\mathbb{Z}}) \cdot (\text{the function sin})))'_{\restriction Z}(x) = n \cdot (\text{the function sin})(x)^{n-1}_{\mathbb{Z}} \cdot (\text{the function cos})((n+1) \cdot x)$.

(12) Suppose $Z \subseteq \mathrm{dom}(((\text{the function cos}) \cdot f)\,((\binom{n}{\mathbb{Z}}) \cdot (\text{the function cos})))$ and $n \geq 1$ and for every x such that $x \in Z$ holds $f(x) = n \cdot x$. Then

(i) $((\text{the function cos}) \cdot f)\,((\binom{n}{\mathbb{Z}}) \cdot (\text{the function cos}))$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(((\text{the function cos}) \cdot f)\,((\binom{n}{\mathbb{Z}}) \cdot (\text{the function cos})))'_{\restriction Z}(x) = -n \cdot (\text{the function cos})(x)^{n-1}_{\mathbb{Z}} \cdot (\text{the function sin})((n+1) \cdot x)$.

(13) Suppose $Z \subseteq \mathrm{dom}(((\text{the function sin}) \cdot f)\,((\binom{n}{\mathbb{Z}}) \cdot (\text{the function cos})))$ and $n \geq 1$ and for every x such that $x \in Z$ holds $f(x) = n \cdot x$. Then

(i) $((\text{the function sin}) \cdot f)\,((\binom{n}{\mathbb{Z}}) \cdot (\text{the function cos}))$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(((\text{the function sin}) \cdot f)\,((\binom{n}{\mathbb{Z}}) \cdot (\text{the function cos})))'_{\restriction Z}(x) = n \cdot (\text{the function cos})(x)^{n-1}_{\mathbb{Z}} \cdot (\text{the function cos})((n+1) \cdot x)$.

(14) Suppose $Z \subseteq \mathrm{dom}(\frac{1}{f}\,(\text{the function sin}))$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) $\frac{1}{f}\,(\text{the function sin})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\frac{1}{f}\,(\text{the function sin}))'_{\restriction Z}(x) = \frac{1}{x} \cdot (\text{the function cos})(x) - \frac{1}{x^{\mathbf{2}}} \cdot (\text{the function sin})(x)$.

(15) Suppose $Z \subseteq \mathrm{dom}(\frac{1}{f}\,(\text{the function cos}))$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) $\frac{1}{f}\,(\text{the function cos})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\frac{1}{f}\,(\text{the function cos}))'_{\restriction Z}(x) = -\frac{1}{x} \cdot (\text{the function sin})(x) - \frac{1}{x^{\mathbf{2}}} \cdot (\text{the function cos})(x)$.

(16) Suppose $Z \subseteq \mathrm{dom}((\text{the function sin}) + (\binom{\frac{1}{2}}{\mathbb{R}}) \cdot f)$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) > 0$. Then

(i) $(\text{the function sin}) + (\binom{\frac{1}{2}}{\mathbb{R}}) \cdot f$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function sin}) + (\binom{\frac{1}{2}}{\mathbb{R}}) \cdot f)'_{\restriction Z}(x) = (\text{the function cos})(x) + \frac{1}{2} \cdot x^{-\frac{1}{2}}_{\mathbb{R}}$.

(17) Suppose $Z \subseteq \mathrm{dom}(g\,((\text{the function sin}) \cdot \frac{1}{f}))$ and $g = \frac{2}{\mathbb{Z}}$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) $g\,((\text{the function sin}) \cdot \frac{1}{f})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(g\,((\text{the function sin}) \cdot \frac{1}{f}))'_{\restriction Z}(x) = 2 \cdot x \cdot (\text{the function sin})(\frac{1}{x}) - (\text{the function cos})(\frac{1}{x})$.

(18) Suppose $Z \subseteq \operatorname{dom}(g\,((\text{the function cos}) \cdot \frac{1}{f}))$ and $g = \frac{2}{\mathbb{Z}}$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) $g\,((\text{the function cos}) \cdot \frac{1}{f})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(g\,((\text{the function cos}) \cdot \frac{1}{f}))'_{\restriction Z}(x) = 2 \cdot x \cdot (\text{the function cos})(\frac{1}{x}) + (\text{the function sin})(\frac{1}{x})$.

(19) Suppose $Z \subseteq \operatorname{dom}(\log_{_}(e) \cdot f)$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) > 0$. Then $\log_{_}(e) \cdot f$ is differentiable on Z and for every x such that $x \in Z$ holds $(\log_{_}(e) \cdot f)'_{\restriction Z}(x) = \frac{1}{x}$.

(20) Suppose $Z \subseteq \operatorname{dom}(\operatorname{id}_Z f)$ and $f = \log_{_}(e) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x$ and $f_1(x) > 0$. Then $\operatorname{id}_Z f$ is differentiable on Z and for every x such that $x \in Z$ holds $(\operatorname{id}_Z f)'_{\restriction Z}(x) = 1 + (\log_{_}(e))(x)$.

(21) Suppose $Z \subseteq \operatorname{dom}(g\,f)$ and $g = \frac{2}{\mathbb{Z}}$ and $f = \log_{_}(e) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x$ and $f_1(x) > 0$. Then $g\,f$ is differentiable on Z and for every x such that $x \in Z$ holds $(g\,f)'_{\restriction Z}(x) = x + 2 \cdot x \cdot (\log_{_}(e))(x)$.

(22) Suppose $Z \subseteq \operatorname{dom}(\frac{f_1+f_2}{f_1-f_2})$ and for every x such that $x \in Z$ holds $f_1(x) = a$ and $f_2 = \frac{2}{\mathbb{Z}}$ and for every x such that $x \in Z$ holds $(f_1 - f_2)(x) > 0$. Then $\frac{f_1+f_2}{f_1-f_2}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{f_1+f_2}{f_1-f_2})'_{\restriction Z}(x) = \frac{4 \cdot a \cdot x}{(a-x^2)^2}$.

(23) Suppose that

(i) $Z \subseteq \operatorname{dom}(\log_{_}(e) \cdot \frac{f_1+f_2}{f_1-f_2})$,

(ii) for every x such that $x \in Z$ holds $f_1(x) = a$,

(iii) $f_2 = \frac{2}{\mathbb{Z}}$,

(iv) for every x such that $x \in Z$ holds $(f_1 - f_2)(x) > 0$, and

(v) for every x such that $x \in Z$ holds $(f_1 + f_2)(x) > 0$.

Then $\log_{_}(e) \cdot \frac{f_1+f_2}{f_1-f_2}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\log_{_}(e) \cdot \frac{f_1+f_2}{f_1-f_2})'_{\restriction Z}(x) = \frac{4 \cdot a \cdot x}{a^2-x^4}$.

(24) Suppose $Z \subseteq \operatorname{dom}(\frac{1}{f}\,g)$ and for every x such that $x \in Z$ holds $f(x) = x$ and $g = \log_{_}(e) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x$ and $f_1(x) > 0$. Then $\frac{1}{f}\,g$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{1}{f}\,g)'_{\restriction Z}(x) = \frac{1}{x^2} \cdot (1 - (\log_{_}(e))(x))$.

(25) Suppose $Z \subseteq \operatorname{dom}(\frac{1}{f})$ and $f = \log_{_}(e) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x$ and $f_1(x) > 0$ and for every x such that $x \in Z$ holds $f(x) \neq 0$. Then $\frac{1}{f}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{1}{f})'_{\restriction Z}(x) = -\frac{1}{x \cdot (\log_{_}(e))(x)^2}$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[2] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[3] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[4] Jarosław Kotowicz. Partial functions from a domain to a domain. *Formalized Mathematics*, 1(**4**):697–702, 1990.
[5] Jarosław Kotowicz. Partial functions from a domain to the set of real numbers. *Formalized Mathematics*, 1(**4**):703–709, 1990.
[6] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[7] Konrad Raczkowski. Integer and rational exponents. *Formalized Mathematics*, 2(**1**):125–130, 1991.
[8] Konrad Raczkowski and Andrzej Nędzusiak. Real exponents and logarithms. *Formalized Mathematics*, 2(**2**):213–216, 1991.
[9] Konrad Raczkowski and Paweł Sadowski. Real function differentiability. *Formalized Mathematics*, 1(**4**):797–801, 1990.
[10] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.
[11] Yasunari Shidama. The Taylor expansions. *Formalized Mathematics*, 12(**2**):195–200, 2004.
[12] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics*.
[13] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[14] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.
[15] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[16] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[17] Yuguang Yang and Yasunari Shidama. Trigonometric functions and existence of circle ratio. *Formalized Mathematics*, 7(**2**):255–263, 1998.

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