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Basic Notions and Properties of Orthoposets[1]

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Summary. Orthoposets are defined. The approach is the standard one via order relation similar to common text books on algebra like [8].

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The terminology and notation used in this paper are introduced in the following papers: [11], [13], [5], [3], [4], [15], [14], [16], [12], [9], [7], [10], [2], [6], and [1].

1. General Notions and Properties

In this paper S, X denote non empty sets and R denotes a binary relation on X.

We consider orthorelational structures, extensions of relational structure and ComplStr, as systems

$\langle$ a carrier, an internal relation, a complement operation $\rangle$,

where the carrier is a set, the internal relation is a binary relation on the carrier, and the complement operation is a unary operation on the carrier.

Let A, B be sets. The functor $\emptyset_{A,B}$ yields a relation between A and B and is defined as follows:

(Def. 1) $\emptyset_{A,B} = \emptyset$.

The functor $\Omega_B(A)$ yields a relation between A and B and is defined by:

(Def. 2) $\Omega_B(A) = [\!: A, B :\!]$.

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We now state several propositions:

(1) $\mathrm{field}(\mathrm{id}_X) = X$.

(2) $\mathrm{id}_{\{\emptyset\}} = \{\langle\emptyset, \emptyset\rangle\}$.

(3) $\mathrm{op}_1 = \{\langle\emptyset, \emptyset\rangle\}$.

(4) Let L be a non empty reflexive antisymmetric relational structure and x, y be elements of L. If $x \leqslant y$, then $\sup\{x, y\} = y$ and $\inf\{x, y\} = x$.

(5) $\mathrm{dom}\, R \subseteq \mathrm{field}\, R$ and $\mathrm{rng}\, R \subseteq \mathrm{field}\, R$.

(6) For all sets A, B holds $\mathrm{field}(\emptyset_{A,B}) = \emptyset$.

Let Y be a set. Note that there exists a binary relation on Y which is antisymmetric.

We now state a number of propositions:

(7) If R is reflexive in X, then R is reflexive and $\mathrm{field}\, R = X$.

(8) If R is symmetric in X, then R is symmetric.

(9) If R is symmetric and $\mathrm{field}\, R \subseteq S$, then R is symmetric in S.

(10) If R is antisymmetric and $\mathrm{field}\, R \subseteq S$, then R is antisymmetric in S.

(11) If R is antisymmetric in X, then R is antisymmetric.

(12) If R is transitive and $\mathrm{field}\, R \subseteq S$, then R is transitive in S.

(13) If R is transitive in X, then R is transitive.

(14) If R is asymmetric and $\mathrm{field}\, R \subseteq S$, then R is asymmetric in S.

(15) If R is asymmetric in X, then R is asymmetric.

(16) If R is irreflexive and $\mathrm{field}\, R \subseteq S$, then R is irreflexive in S.

(17) If R is irreflexive in X, then R is irreflexive.

Let X be a set. Observe that every binary relation on X which is equivalence relation-like is also reflexive, symmetric, and transitive.

Let us consider X. One can check that there exists a binary relation on X which is equivalence relation-like.

Let X be a set. Note that there exists a binary relation on X which is irreflexive, asymmetric, and transitive.

The following proposition is true

(18) $\triangle_\emptyset$ is antisymmetric.

Let us consider X, R and let C be a unary operation on X. Note that $\langle X, R, C\rangle$ is non empty.

Let us mention that there exists a orthorelational structure which is non empty and strict.

Let us consider X and let f be a unary operation on X. We say that f is dneg if and only if:

(Def. 3) For every element x of X holds $f(f(x)) = x$.

We introduce f is involutive as a synonym of f is dneg.

One can prove the following two propositions:

(19) op_1 is dneg.

(20) id_X is dneg.

Let O be a non empty orthorelational structure and let f be a map from O into O. We say that f is DNeg if and only if:

(Def. 4) f is dneg.

Let O be a non empty orthorelational structure. Observe that there exists a map from O into O which is DNeg.

The strict orthorelational structure TrivOrthoRelStr is defined as follows:

(Def. 5) TrivOrthoRelStr $= \langle \{\emptyset\}, \mathrm{id}_{\{\emptyset\}}, \mathrm{op}_1 \rangle$.

We introduce TrivPoset as a synonym of TrivOrthoRelStr.

Let us mention that TrivOrthoRelStr is non empty.

The strict orthorelational structure TrivAsymOrthoRelStr is defined by:

(Def. 6) TrivAsymOrthoRelStr $= \langle \{\emptyset\}, \emptyset_{\{\emptyset\},\{\emptyset\}}, \mathrm{op}_1 \rangle$.

Let us mention that TrivAsymOrthoRelStr is non empty.

Let O be a non empty orthorelational structure. We say that O is Dneg if and only if:

(Def. 7) There exists a map f from O into O such that $f =$ the complement operation of O and f is DNeg.

One can prove the following proposition

(21) TrivOrthoRelStr is Dneg.

Let us note that TrivOrthoRelStr is Dneg.

Let us observe that there exists a non empty orthorelational structure which is Dneg.

In the sequel O is a non empty orthorelational structure.

Let R_1, R_2 be relational structures and let f be a map from R_1 into R_2. We say that f is Antitone on R_1, R_2 if and only if:

(Def. 8) f is antitone.

Let R be a relational structure and let f be a map from R into R. We say that f is Antitone on R if and only if:

(Def. 9) f is Antitone on R, R.

Let us consider O. We say that O is SubReFlexive if and only if:

(Def. 10) The internal relation of O is reflexive.

Let us consider O. We say that O is ReFlexive if and only if:

(Def. 11) The internal relation of O is reflexive in the carrier of O.

We now state two propositions:

(22) If O is ReFlexive, then O is SubReFlexive.

(23) TrivOrthoRelStr is ReFlexive.

Let us observe that TrivOrthoRelStr is ReFlexive.

One can verify that there exists a non empty orthorelational structure which is ReFlexive and strict.

Let us consider O. We say that O is SubIrreFlexive if and only if:

(Def. 12) The internal relation of O is irreflexive.

We say that O is IrreFlexive if and only if:

(Def. 13) The internal relation of O is irreflexive in the carrier of O.

We now state two propositions:

(24) If O is IrreFlexive, then O is SubIrreFlexive.

(25) TrivAsymOrthoRelStr is IrreFlexive.

Let us note that every non empty orthorelational structure which is IrreFlexive is also SubIrreFlexive.

Let us observe that TrivAsymOrthoRelStr is IrreFlexive.

Let us note that there exists a non empty orthorelational structure which is IrreFlexive and strict.

Let us consider O. We say that O is SubSymmetric if and only if:

(Def. 14) The internal relation of O is a symmetric binary relation on the carrier of O.

Let us consider O. We say that O is Symmetric if and only if:

(Def. 15) The internal relation of O is symmetric in the carrier of O.

We now state two propositions:

(26) If O is Symmetric, then O is SubSymmetric.

(27) TrivOrthoRelStr is Symmetric.

Let us observe that every non empty orthorelational structure which is Symmetric is also SubSymmetric.

Let us note that there exists a non empty orthorelational structure which is Symmetric and strict.

Let us consider O. We say that O is SubAntisymmetric if and only if:

(Def. 16) The internal relation of O is an antisymmetric binary relation on the carrier of O.

Let us consider O. We say that O is Antisymmetric if and only if:

(Def. 17) The internal relation of O is antisymmetric in the carrier of O.

Next we state two propositions:

(28) If O is Antisymmetric, then O is SubAntisymmetric.

(29) TrivOrthoRelStr is Antisymmetric.

Let us observe that every non empty orthorelational structure which is Antisymmetric is also SubAntisymmetric.

One can verify that TrivOrthoRelStr is Symmetric and Antisymmetric.

One can check that there exists a non empty orthorelational structure which is Symmetric, Antisymmetric, and strict.

Let us consider O. We say that O is SubAsymmetric if and only if:

(Def. 18) The internal relation of O is an asymmetric binary relation on the carrier of O.

Let us consider O. We say that O is Asymmetric if and only if:

(Def. 19) The internal relation of O is asymmetric in the carrier of O.

One can prove the following two propositions:

(30) If O is Asymmetric, then O is SubAsymmetric.

(31) TrivAsymOrthoRelStr is Asymmetric.

Let us mention that every non empty orthorelational structure which is Asymmetric is also SubAsymmetric.

One can check that TrivAsymOrthoRelStr is Asymmetric.

Let us observe that there exists a non empty orthorelational structure which is Asymmetric and strict.

Let us consider O. We say that O is SubTransitive if and only if:

(Def. 20) The internal relation of O is a transitive binary relation on the carrier of O.

Let us consider O. We say that O is Transitive if and only if:

(Def. 21) The internal relation of O is transitive in the carrier of O.

Next we state two propositions:

(32) If O is Transitive, then O is SubTransitive.

(33) TrivOrthoRelStr is Transitive.

Let us observe that every non empty orthorelational structure which is Transitive is also SubTransitive.

Let us observe that TrivOrthoRelStr is Transitive.

Let us observe that there exists a non empty orthorelational structure which is ReFlexive, Symmetric, Antisymmetric, Transitive, and strict.

Next we state the proposition

(34) TrivAsymOrthoRelStr is Transitive.

Let us mention that TrivAsymOrthoRelStr is IrreFlexive, Asymmetric, and Transitive.

Let us observe that there exists a non empty orthorelational structure which is IrreFlexive, Asymmetric, Transitive, and strict.

Next we state four propositions:

(35) If O is SubSymmetric and SubTransitive, then O is SubReFlexive.

(36) If O is SubIrreFlexive and SubTransitive, then O is SubAsymmetric.

(37) If O is SubAsymmetric, then O is SubIrreFlexive.

(38) If O is ReFlexive and SubSymmetric, then O is Symmetric.

One can check that every non empty orthorelational structure which is ReFlexive and SubSymmetric is also Symmetric.

Next we state the proposition

(39) If O is ReFlexive and SubAntisymmetric, then O is Antisymmetric.

Let us note that every non empty orthorelational structure which is ReFlexive and SubAntisymmetric is also Antisymmetric.

The following proposition is true

(40) If O is ReFlexive and SubTransitive, then O is Transitive.

Let us note that every non empty orthorelational structure which is ReFlexive and SubTransitive is also Transitive.

One can prove the following proposition

(41) If O is IrreFlexive and SubTransitive, then O is Transitive.

Let us observe that every non empty orthorelational structure which is IrreFlexive and SubTransitive is also Transitive.

Next we state the proposition

(42) If O is IrreFlexive and SubAsymmetric, then O is Asymmetric.

Let us note that every non empty orthorelational structure which is IrreFlexive and SubAsymmetric is also Asymmetric.

2. Basic Poset Notions

Let us consider O. We say that O is SubQuasiOrdered if and only if:

(Def. 22) O is SubReFlexive and SubTransitive.

We introduce O is SubQuasiordered, O is SubPreOrdered, O is SubPreordered, and O is Subpreordered as synonyms of O is SubQuasiOrdered.

Let us consider O. We say that O is QuasiOrdered if and only if:

(Def. 23) O is ReFlexive and Transitive.

We introduce O is Quasiordered, O is PreOrdered, and O is Preordered as synonyms of O is QuasiOrdered.

The following proposition is true

(43) If O is QuasiOrdered, then O is SubQuasiOrdered.

Let us observe that every non empty orthorelational structure which is QuasiOrdered is also SubQuasiOrdered.

Let us note that TrivOrthoRelStr is QuasiOrdered.

Let us consider O. We say that O is QuasiPure if and only if:

(Def. 24) O is Dneg and QuasiOrdered.

Let us mention that there exists a non empty orthorelational structure which is QuasiPure, Dneg, QuasiOrdered, and strict.

Let us note that TrivOrthoRelStr is QuasiPure.

A QuasiPureOrthoRelStr is a QuasiPure non empty orthorelational structure.

Let us consider O. We say that O is SubPartialOrdered if and only if:

(Def. 25) O is ReFlexive, SubAntisymmetric, and SubTransitive.

We introduce O is SubPartialordered as a synonym of O is SubPartialOrdered.

Let us consider O. We say that O is PartialOrdered if and only if:

(Def. 26) O is ReFlexive, Antisymmetric, and Transitive.

We introduce O is Partialordered as a synonym of O is PartialOrdered.

We now state the proposition

(44) O is SubPartialOrdered iff O is PartialOrdered.

Let us note that every non empty orthorelational structure which is SubPartialOrdered is also PartialOrdered and every non empty orthorelational structure which is PartialOrdered is also SubPartialOrdered.

Let us observe that every non empty orthorelational structure which is PartialOrdered is also ReFlexive, Antisymmetric, and Transitive and every non empty orthorelational structure which is ReFlexive, Antisymmetric, and Transitive is also PartialOrdered.

Let us consider O. We say that O is Pure if and only if:

(Def. 27) O is Dneg and PartialOrdered.

Let us mention that there exists a non empty orthorelational structure which is Pure, Dneg, PartialOrdered, and strict.

One can check that TrivOrthoRelStr is Pure.

A PureOrthoRelStr is a Pure non empty orthorelational structure.

Let us consider O. We say that O is SubStrictPartialOrdered if and only if:

(Def. 28) O is SubAsymmetric and SubTransitive.

Let us consider O. We say that O is StrictPartialOrdered if and only if:

(Def. 29) O is Asymmetric and Transitive.

We introduce O is Strictpartialordered, O is StrictOrdered, and O is Strictordered as synonyms of O is StrictPartialOrdered.

The following proposition is true

(45) If O is StrictPartialOrdered, then O is SubStrictPartialOrdered.

Let us note that every non empty orthorelational structure which is StrictPartialOrdered is also SubStrictPartialOrdered.

One can prove the following proposition

(46) If O is SubStrictPartialOrdered, then O is SubIrreFlexive.

Let us note that every non empty orthorelational structure which is SubStrictPartialOrdered is also SubIrreFlexive.

Next we state the proposition

(47) If O is IrreFlexive and SubStrictPartialOrdered, then O is StrictPartialOrdered.

Let us mention that every non empty orthorelational structure which is IrreFlexive and SubStrictPartialOrdered is also StrictPartialOrdered.

We now state the proposition

(48) If O is StrictPartialOrdered, then O is IrreFlexive.

Let us note that every non empty orthorelational structure which is StrictPartialOrdered is also IrreFlexive.

One can check that TrivAsymOrthoRelStr is IrreFlexive and StrictPartialOrdered.

Let us mention that there exists a non empty strict orthorelational structure which is IrreFlexive and StrictPartialOrdered.

In the sequel P_1 denotes a PartialOrdered non empty orthorelational structure and Q_1 denotes a QuasiOrdered non empty orthorelational structure.

We now state the proposition

(49) If Q_1 is SubAntisymmetric, then Q_1 is PartialOrdered.

Let P_1 be a PartialOrdered non empty orthorelational structure. Note that the internal relation of P_1 is ordering.

One can prove the following proposition

(50) P_1 is a poset.

Let us note that every non empty orthorelational structure which is PartialOrdered is also reflexive, transitive, and antisymmetric.

Let P_2, P_3 be PartialOrdered non empty orthorelational structures and let f be a map from P_2 into P_3. We say that f is Antitone on P_2, P_3 if and only if:

(Def. 30) f is antitone.

Let P_1 be a PartialOrdered non empty orthorelational structure and let f be a map from P_1 into P_1. We say that f is Antitone on P_1 if and only if:

(Def. 31) f is Antitone on P_1, P_1.

Let P_2, P_3 be PartialOrdered non empty orthorelational structures and let f be a map from P_2 into P_3. We say that f is Antitone if and only if:

(Def. 32) f is Antitone on P_2, P_3.

Let P_1 be a PartialOrdered non empty orthorelational structure. Note that there exists a map from P_1 into P_1 which is Antitone.

Let us consider P_1 and let f be a unary operation on the carrier of P_1. We say that f is Orderinvolutive if and only if:

(Def. 33) f is a DNeg map from P_1 into P_1 and an Antitone map from P_1 into P_1.

Let us consider P_1. We say that P_1 is OrderInvolutive if and only if:

(Def. 34) There exists a map f from P_1 into P_1 such that $f =$ the complement operation of P_1 and f is Orderinvolutive.

Next we state the proposition

(51) The complement operation of TrivOrthoRelStr is Orderinvolutive.

Let us observe that TrivOrthoRelStr is OrderInvolutive.

One can check that there exists a PartialOrdered non empty orthorelational structure which is OrderInvolutive and Pure.

A PreOrthoPoset is an OrderInvolutive Pure PartialOrdered non empty orthorelational structure.

Let us consider P_1 and let f be a unary operation on the carrier of P_1. We say that f is QuasiOrthoComplement on P_1 if and only if:

(Def. 35) f is Orderinvolutive and for every element y of P_1 holds $\sup\{y, f(y)\}$ exists in P_1 and $\inf\{y, f(y)\}$ exists in P_1.

Let us consider P_1. We say that P_1 is QuasiOrthocomplemented if and only if:

(Def. 36) There exists a map f from P_1 into P_1 such that $f =$ the complement operation of P_1 and f is QuasiOrthoComplement on P_1.

Next we state the proposition

(52) TrivOrthoRelStr is QuasiOrthocomplemented.

Let us consider P_1 and let f be a unary operation on the carrier of P_1. We say that f is OrthoComplement on P_1 if and only if the conditions (Def. 37) are satisfied.

(Def. 37)(i) f is Orderinvolutive, and

(ii) for every element y of P_1 holds $\sup\{y, f(y)\}$ exists in P_1 and $\inf\{y, f(y)\}$ exists in P_1 and $\bigsqcup_{P_1}\{y, f(y)\}$ is a maximum of the carrier of P_1 and $\bigsqcap_{P_1}\{y, f(y)\}$ is a minimum of the carrier of P_1.

We introduce f is OCompl on P_1 as a synonym of f is OrthoComplement on P_1.

Let us consider P_1. We say that P_1 is Orthocomplemented if and only if:

(Def. 38) There exists a map f from P_1 into P_1 such that $f =$ the complement operation of P_1 and f is OrthoComplement on P_1.

We introduce P_1 is Ocompl as a synonym of P_1 is Orthocomplemented.

Next we state two propositions:

(53) Let f be a unary operation on the carrier of P_1. If f is OrthoComplement on P_1, then f is QuasiOrthoComplement on P_1.

(54) TrivOrthoRelStr is Orthocomplemented.

One can check that TrivOrthoRelStr is QuasiOrthocomplemented and Orthocomplemented.

Let us mention that there exists a PartialOrdered non empty orthorelational structure which is Orthocomplemented and QuasiOrthocomplemented.

A QuasiOrthoPoset is a QuasiOrthocomplemented PartialOrdered non empty orthorelational structure. An orthoposet is an Orthocomplemented PartialOrdered non empty orthorelational structure.

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