FORMALIZED MATHEMATICS
Volume 12, Number 2, 2004
University of Białystok

# Complex Banach Space of Bounded Complex Sequences

Noboru Endou
Gifu National College of Technology

**Summary.** An extension of [18]. In this article, we introduce two complex Banach spaces. One of them is the space of bounded complex sequences. The other one is the space of complex bounded functions, which is defined by the set of all complex bounded functions.

MML Identifier: CSSPACE4.

The articles [21], [6], [23], [24], [17], [20], [2], [19], [12], [4], [5], [7], [22], [3], [1], [16], [15], [14], [10], [13], [11], [8], and [9] provide the terminology and notation for this paper.

## 1. Complex Banach Space of Bounded Complex Sequences

The subset the set of bounded complex sequences of the linear space of complex sequences is defined by the condition (Def. 1).

(Def. 1) Let $x$ be a set. Then $x \in$ the set of bounded complex sequences if and only if $x \in$ the set of complex sequences and $\mathrm{id}_{\mathrm{seq}}(x)$ is bounded.

Let us note that the set of bounded complex sequences is non empty and the set of bounded complex sequences is linearly closed.

One can prove the following proposition

(1) $\langle$the set of bounded complex sequences, Zero_(the set of bounded complex sequences, the linear space of complex sequences), Add_(the set of bounded complex sequences, the linear space of complex sequences), Mult_(the set of bounded complex sequences, the linear space of complex sequences)$\rangle$ is a subspace of the linear space of complex sequences.



© 2004 University of Białystok
ISSN 1426–2630

Let us mention that ⟨the set of bounded complex sequences, Zero_(the set of bounded complex sequences, the linear space of complex sequences), Add_(the set of bounded complex sequences, the linear space of complex sequences),

Mult_(the set of bounded complex sequences, the linear space of complex sequences)⟩ is Abelian, add-associative, right zeroed, right complementable, and complex linear space-like.

The function Clinfty-norm from the set of bounded complex sequences into $\mathbb{R}$ is defined by:

(Def. 2) For every set $x$ such that $x \in$ the set of bounded complex sequences holds Clinfty-norm$(x) = \sup \operatorname{rng} |\operatorname{id}_{\text{seq}}(x)|$.

Next we state the proposition

(2) For every complex sequence $s_1$ holds $s_1$ is bounded and $\sup \operatorname{rng} |s_1| = 0$ iff for every natural number $n$ holds $s_1(n) = 0_{\mathbb{C}}$.

One can check that ⟨the set of bounded complex sequences, Zero_(the set of bounded complex sequences, the linear space of complex sequences), Add_(the set of bounded complex sequences, the linear space of complex sequences),

Mult_(the set of bounded complex sequences, the linear space of complex sequences), Clinfty-norm⟩ is Abelian, add-associative, right zeroed, right complementable, and complex linear space-like.

The non empty complex normed space structure Clinfty-Space is defined by the condition (Def. 3).

(Def. 3) Clinfty-Space = ⟨the set of bounded complex sequences, Zero_(the set of bounded complex sequences, the linear space of complex sequences), Add_(the set of bounded complex sequences, the linear space of complex sequences), Mult_(the set of bounded complex sequences, the linear space of complex sequences), Clinfty-norm⟩.

Next we state two propositions:

(3) The carrier of Clinfty-Space = the set of bounded complex sequences and for every set $x$ holds $x$ is a vector of Clinfty-Space iff $x$ is a complex sequence and $\operatorname{id}_{\text{seq}}(x)$ is bounded and $0_{\text{Clinfty-Space}} =$ CZeroseq and for every vector $u$ of Clinfty-Space holds $u = \operatorname{id}_{\text{seq}}(u)$ and for all vectors $u$, $v$ of Clinfty-Space holds $u + v = \operatorname{id}_{\text{seq}}(u) + \operatorname{id}_{\text{seq}}(v)$ and for every Complex $c$ and for every vector $u$ of Clinfty-Space holds $c \cdot u = c \operatorname{id}_{\text{seq}}(u)$ and for every vector $u$ of Clinfty-Space holds $-u = -\operatorname{id}_{\text{seq}}(u)$ and $\operatorname{id}_{\text{seq}}(-u) = -\operatorname{id}_{\text{seq}}(u)$ and for all vectors $u$, $v$ of Clinfty-Space holds $u - v = \operatorname{id}_{\text{seq}}(u) - \operatorname{id}_{\text{seq}}(v)$ and for every vector $v$ of Clinfty-Space holds $\operatorname{id}_{\text{seq}}(v)$ is bounded and for every vector $v$ of Clinfty-Space holds $\|v\| = \sup \operatorname{rng} |\operatorname{id}_{\text{seq}}(v)|$.

(4) Let $x$, $y$ be points of Clinfty-Space and $c$ be a Complex. Then $\|x\| = 0$ iff $x = 0_{\text{Clinfty-Space}}$ and $0 \leqslant \|x\|$ and $\|x+y\| \leqslant \|x\|+\|y\|$ and $\|c \cdot x\| = |c| \cdot \|x\|$.

Let us note that Clinfty-Space is complex normed space-like, complex linear

space-like, Abelian, add-associative, right zeroed, and right complementable.

We now state two propositions:

(5) For every sequence $v_1$ of Clinfty-Space such that $v_1$ is Cauchy sequence by norm holds $v_1$ is convergent.

(6) Clinfty-Space is a complex Banach space.

## 2. Another Example of Complex Banach Space

Let $X$ be a non empty set, let $Y$ be a complex normed space, and let $I_1$ be a function from $X$ into the carrier of $Y$. We say that $I_1$ is bounded if and only if:

(Def. 4) There exists a real number $K$ such that $0 \leqslant K$ and for every element $x$ of $X$ holds $\|I_1(x)\| \leqslant K$.

The following proposition is true

(7) Let $X$ be a non empty set, $Y$ be a complex normed space, and $f$ be a function from $X$ into the carrier of $Y$. If for every element $x$ of $X$ holds $f(x) = 0_Y$, then $f$ is bounded.

Let $X$ be a non empty set and let $Y$ be a complex normed space. One can check that there exists a function from $X$ into the carrier of $Y$ which is bounded.

Let $X$ be a non empty set and let $Y$ be a complex normed space. The functor $\mathrm{CBdFuncs}(X, Y)$ yields a subset of $\mathrm{ComplexVectSpace}(X, Y)$ and is defined by:

(Def. 5) For every set $x$ holds $x \in \mathrm{CBdFuncs}(X, Y)$ iff $x$ is a bounded function from $X$ into the carrier of $Y$.

Let $X$ be a non empty set and let $Y$ be a complex normed space. Note that $\mathrm{CBdFuncs}(X, Y)$ is non empty.

One can prove the following propositions:

(8) For every non empty set $X$ and for every complex normed space $Y$ holds $\mathrm{CBdFuncs}(X, Y)$ is linearly closed.

(9) Let $X$ be a non empty set and $Y$ be a complex normed space. Then $\langle \mathrm{CBdFuncs}(X, Y), \mathrm{Zero_}(\mathrm{CBdFuncs}(X, Y), \mathrm{ComplexVectSpace}(X, Y)), \mathrm{Add_}(\mathrm{CBdFuncs}(X, Y), \mathrm{ComplexVectSpace}(X, Y)), \mathrm{Mult_}(\mathrm{CBdFuncs}(X, Y), \mathrm{ComplexVectSpace}(X, Y))\rangle$ is a subspace of $\mathrm{ComplexVectSpace}(X, Y)$.

Let $X$ be a non empty set and let $Y$ be a complex normed space. Note that $\langle \mathrm{CBdFuncs}(X, Y), \mathrm{Zero_}(\mathrm{CBdFuncs}(X, Y), \mathrm{ComplexVectSpace}(X, Y)), \mathrm{Add_}(\mathrm{CBdFuncs}(X, Y), \mathrm{ComplexVectSpace}(X, Y)), \mathrm{Mult_}(\mathrm{CBdFuncs}(X, Y), \mathrm{ComplexVectSpace}(X, Y))\rangle$ is Abelian, add-associative, right zeroed, right complementable, and complex linear space-like.

We now state the proposition

(10) Let $X$ be a non empty set and $Y$ be a complex normed space. Then $\langle \mathrm{CBdFuncs}(X,Y), \mathrm{Zero_}(\mathrm{CBdFuncs}(X,Y), \mathrm{ComplexVectSpace}(X,Y)), \mathrm{Add_}(\mathrm{CBdFuncs}(X,Y), \mathrm{ComplexVectSpace}(X,Y)), \mathrm{Mult_}(\mathrm{CBdFuncs}(X,Y), \mathrm{ComplexVectSpace}(X,Y))\rangle$ is a complex linear space.

Let $X$ be a non empty set and let $Y$ be a complex normed space. The set of bounded complex sequences from $X$ into $Y$ yielding a complex linear space is defined by:

(Def. 6) The set of bounded complex sequences from $X$ into $Y$ $=$ $\langle \mathrm{CBdFuncs}(X,Y), \mathrm{Zero_}(\mathrm{CBdFuncs}(X,Y), \mathrm{ComplexVectSpace}(X,Y)), \mathrm{Add_}(\mathrm{CBdFuncs}(X,Y), \mathrm{ComplexVectSpace}(X,Y)), \mathrm{Mult_}(\mathrm{CBdFuncs}(X,Y), \mathrm{ComplexVectSpace}(X,Y))\rangle$.

Let $X$ be a non empty set and let $Y$ be a complex normed space. One can verify that the set of bounded complex sequences from $X$ into $Y$ is strict.

The following three propositions are true:

(11) Let $X$ be a non empty set, $Y$ be a complex normed space, $f$, $g$, $h$ be vectors of the set of bounded complex sequences from $X$ into $Y$, and $f'$, $g'$, $h'$ be bounded functions from $X$ into the carrier of $Y$. Suppose $f' = f$ and $g' = g$ and $h' = h$. Then $h = f + g$ if and only if for every element $x$ of $X$ holds $h'(x) = f'(x) + g'(x)$.

(12) Let $X$ be a non empty set, $Y$ be a complex normed space, $f$, $h$ be vectors of the set of bounded complex sequences from $X$ into $Y$, and $f'$, $h'$ be bounded functions from $X$ into the carrier of $Y$. Suppose $f' = f$ and $h' = h$. Let $c$ be a Complex. Then $h = c \cdot f$ if and only if for every element $x$ of $X$ holds $h'(x) = c \cdot f'(x)$.

(13) Let $X$ be a non empty set and $Y$ be a complex normed space. Then $0_{\text{the set of bounded complex sequences from } X \text{ into } Y} = X \longmapsto 0_Y$.

Let $X$ be a non empty set, let $Y$ be a complex normed space, and let $f$ be a set. Let us assume that $f \in \mathrm{CBdFuncs}(X,Y)$. The functor $\mathrm{modetrans}(f,X,Y)$ yields a bounded function from $X$ into the carrier of $Y$ and is defined by:

(Def. 7) $\mathrm{modetrans}(f,X,Y) = f$.

Let $X$ be a non empty set, let $Y$ be a complex normed space, and let $u$ be a function from $X$ into the carrier of $Y$. The functor $\mathrm{PreNorms}(u)$ yielding a non empty subset of $\mathbb{R}$ is defined by:

(Def. 8) $\mathrm{PreNorms}(u) = \{\|u(t)\| : t \text{ ranges over elements of } X\}$.

We now state three propositions:

(14) Let $X$ be a non empty set, $Y$ be a complex normed space, and $g$ be a bounded function from $X$ into the carrier of $Y$. Then $\mathrm{PreNorms}(g)$ is non empty and upper bounded.

(15) Let $X$ be a non empty set, $Y$ be a complex normed space, and $g$ be a function from $X$ into the carrier of $Y$. Then $g$ is bounded if and only if

PreNorms($g$) is upper bounded.

(16) Let $X$ be a non empty set and $Y$ be a complex normed space. Then there exists a function $N_1$ from CBdFuncs($X, Y$) into $\mathbb{R}$ such that for every set $f$ if $f \in$ CBdFuncs($X, Y$), then $N_1(f) =$ sup PreNorms(modetrans($f, X, Y$)).

Let $X$ be a non empty set and let $Y$ be a complex normed space. The functor CBdFuncsNorm($X, Y$) yielding a function from CBdFuncs($X, Y$) into $\mathbb{R}$ is defined by:

(Def. 9) For every set $x$ such that $x \in$ CBdFuncs($X, Y$) holds CBdFuncsNorm($X, Y$)($x$) = sup PreNorms(modetrans($x, X, Y$)).

One can prove the following propositions:

(17) Let $X$ be a non empty set, $Y$ be a complex normed space, and $f$ be a bounded function from $X$ into the carrier of $Y$. Then modetrans($f, X, Y$) = $f$.

(18) Let $X$ be a non empty set, $Y$ be a complex normed space, and $f$ be a bounded function from $X$ into the carrier of $Y$. Then CBdFuncsNorm($X, Y$)($f$) = sup PreNorms($f$).

Let $X$ be a non empty set and let $Y$ be a complex normed space. The complex normed space of bounded functions from $X$ into $Y$ yields a non empty complex normed space structure and is defined by:

(Def. 10) The complex normed space of bounded functions from $X$ into $Y$ = $\langle$CBdFuncs($X, Y$), Zero_(CBdFuncs($X, Y$), ComplexVectSpace($X, Y$)), Add_(CBdFuncs($X, Y$), ComplexVectSpace($X, Y$)), Mult_(CBdFuncs($X, Y$), ComplexVectSpace($X, Y$)), CBdFuncsNorm($X, Y$)$\rangle$.

The following propositions are true:

(19) Let $X$ be a non empty set and $Y$ be a complex normed space. Then $X \longmapsto 0_Y = 0_{\text{the complex normed space of bounded functions from } X \text{ into } Y}$.

(20) Let $X$ be a non empty set, $Y$ be a complex normed space, $f$ be a point of the complex normed space of bounded functions from $X$ into $Y$, and $g$ be a bounded function from $X$ into the carrier of $Y$. If $g = f$, then for every element $t$ of $X$ holds $\|g(t)\| \leqslant \|f\|$.

(21) Let $X$ be a non empty set, $Y$ be a complex normed space, and $f$ be a point of the complex normed space of bounded functions from $X$ into $Y$. Then $0 \leqslant \|f\|$.

(22) Let $X$ be a non empty set, $Y$ be a complex normed space, and $f$ be a point of the complex normed space of bounded functions from $X$ into $Y$. Suppose $f = 0_{\text{the complex normed space of bounded functions from } X \text{ into } Y}$. Then $0 = \|f\|$.

(23) Let $X$ be a non empty set, $Y$ be a complex normed space, $f$, $g$, $h$ be points of the complex normed space of bounded functions from $X$ into $Y$,

and $f'$, $g'$, $h'$ be bounded functions from $X$ into the carrier of $Y$. Suppose $f' = f$ and $g' = g$ and $h' = h$. Then $h = f + g$ if and only if for every element $x$ of $X$ holds $h'(x) = f'(x) + g'(x)$.

(24) Let $X$ be a non empty set, $Y$ be a complex normed space, $f$, $h$ be points of the complex normed space of bounded functions from $X$ into $Y$, and $f'$, $h'$ be bounded functions from $X$ into the carrier of $Y$. Suppose $f' = f$ and $h' = h$. Let $c$ be a Complex. Then $h = c \cdot f$ if and only if for every element $x$ of $X$ holds $h'(x) = c \cdot f'(x)$.

(25) Let $X$ be a non empty set, $Y$ be a complex normed space, $f$, $g$ be points of the complex normed space of bounded functions from $X$ into $Y$, and $c$ be a Complex. Then

(i) $\|f\| = 0$ iff $f = 0_{\text{the complex normed space of bounded functions from } X \text{ into } Y}$,

(ii) $\|c \cdot f\| = |c| \cdot \|f\|$, and

(iii) $\|f + g\| \leqslant \|f\| + \|g\|$.

(26) Let $X$ be a non empty set and $Y$ be a complex normed space. Then the complex normed space of bounded functions from $X$ into $Y$ is complex normed space-like.

(27) Let $X$ be a non empty set and $Y$ be a complex normed space. Then the complex normed space of bounded functions from $X$ into $Y$ is a complex normed space.

Let $X$ be a non empty set and let $Y$ be a complex normed space. One can check that the complex normed space of bounded functions from $X$ into $Y$ is complex normed space-like, complex linear space-like, Abelian, add-associative, right zeroed, and right complementable.

One can prove the following three propositions:

(28) Let $X$ be a non empty set, $Y$ be a complex normed space, $f$, $g$, $h$ be points of the complex normed space of bounded functions from $X$ into $Y$, and $f'$, $g'$, $h'$ be bounded functions from $X$ into the carrier of $Y$. Suppose $f' = f$ and $g' = g$ and $h' = h$. Then $h = f - g$ if and only if for every element $x$ of $X$ holds $h'(x) = f'(x) - g'(x)$.

(29) Let $X$ be a non empty set and $Y$ be a complex normed space. Suppose $Y$ is complete. Let $s_1$ be a sequence of the complex normed space of bounded functions from $X$ into $Y$. If $s_1$ is Cauchy sequence by norm, then $s_1$ is convergent.

(30) Let $X$ be a non empty set and $Y$ be a complex Banach space. Then the complex normed space of bounded functions from $X$ into $Y$ is a complex Banach space.

Let $X$ be a non empty set and let $Y$ be a complex Banach space. Note that the complex normed space of bounded functions from $X$ into $Y$ is complete.

## 3. Some Properties of Complex Sequences

We now state four propositions:

(31) For all complex sequences $s_2$, $s_3$ such that $s_2$ is bounded and $s_3$ is bounded holds $s_2 + s_3$ is bounded.

(32) For every Complex $c$ and for every complex sequence $s_1$ such that $s_1$ is bounded holds $c\,s_1$ is bounded.

(33) For every complex sequence $s_1$ holds $s_1$ is bounded iff $|s_1|$ is bounded.

(34) For all complex sequences $s_2$, $s_3$, $s_4$ holds $s_2 = s_3 - s_4$ iff for every natural number $n$ holds $s_2(n) = s_3(n) - s_4(n)$.

## References


[1] Agnieszka Banachowicz and Anna Winnicka. Complex sequences. *Formalized Mathematics*, 4(**1**):121–124, 1993.
[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[3] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(**3**):507–513, 1990.
[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[6] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[7] Czesław Byliński and Piotr Rudnicki. Bounding boxes for compact sets in $\mathcal{E}^2$. *Formalized Mathematics*, 6(**3**):427–440, 1997.
[8] Noboru Endou. Banach space of absolute summable complex sequences. *Formalized Mathematics*, 12(**2**):191–194, 2004.
[9] Noboru Endou. Complex Banach space of bounded linear operators. *Formalized Mathematics*, 12(**2**):201–209, 2004.
[10] Noboru Endou. Complex linear space and complex normed space. *Formalized Mathematics*, 12(**2**):93–102, 2004.
[11] Noboru Endou. Complex linear space of complex sequences. *Formalized Mathematics*, 12(**2**):109–117, 2004.
[12] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[13] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Formalized Mathematics*, 1(**3**):477–481, 1990.
[14] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.
[15] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[16] Adam Naumowicz. Conjugate sequences, bounded complex sequences and convergent complex sequences. *Formalized Mathematics*, 6(**2**):265–268, 1997.
[17] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[18] Yasumasa Suzuki. Banach space of bounded real sequences. *Formalized Mathematics*, 12(**2**):77–83, 2004.
[19] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics*.
[20] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[21] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[22] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.

[23] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[24] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

*Received March 18, 2004*