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Convergent Sequences in Complex Unitary Space

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Summary. In this article, we introduce the notion of convergence sequence in complex unitary space and complex Hilbert space.

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The terminology and notation used in this paper are introduced in the following papers: [15], [2], [14], [7], [1], [17], [3], [4], [10], [9], [16], [13], [11], [12], [8], [5], and [6].

1. Convergence in Complex Unitary Space

For simplicity, we adopt the following convention: X is a complex unitary space, x, y, w, g, g_1, g_2 are points of X, z is a Complex, q, r, M are real numbers, s_1, s_2, s_3, s_4 are sequences of X, k, n, m are natural numbers, and N_1 is an increasing sequence of naturals.

Let us consider X, s_1. We say that s_1 is convergent if and only if:

(Def. 1) There exists g such that for every r such that $r > 0$ there exists m such that for every n such that $n \geqslant m$ holds $\rho(s_1(n), g) < r$.

Next we state several propositions:

(1) If s_1 is constant, then s_1 is convergent.

(2) If s_2 is convergent and there exists k such that for every n such that $k \leqslant n$ holds $s_3(n) = s_2(n)$, then s_3 is convergent.

(3) If s_2 is convergent and s_3 is convergent, then $s_2 + s_3$ is convergent.

(4) If s_2 is convergent and s_3 is convergent, then $s_2 - s_3$ is convergent.

(5) If s_1 is convergent, then $z \cdot s_1$ is convergent.

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(6) If s_1 is convergent, then $-s_1$ is convergent.

(7) If s_1 is convergent, then $s_1 + x$ is convergent.

(8) If s_1 is convergent, then $s_1 - x$ is convergent.

(9) s_1 is convergent if and only if there exists g such that for every r such that $r > 0$ there exists m such that for every n such that $n \geqslant m$ holds $\|s_1(n) - g\| < r$.

Let us consider X, s_1. Let us assume that s_1 is convergent. The functor $\lim s_1$ yields a point of X and is defined as follows:

(Def. 2) For every r such that $r > 0$ there exists m such that for every n such that $n \geqslant m$ holds $\rho(s_1(n), \lim s_1) < r$.

One can prove the following propositions:

(10) If s_1 is constant and $x \in \operatorname{rng} s_1$, then $\lim s_1 = x$.

(11) If s_1 is constant and there exists n such that $s_1(n) = x$, then $\lim s_1 = x$.

(12) If s_2 is convergent and there exists k such that for every n such that $n \geqslant k$ holds $s_3(n) = s_2(n)$, then $\lim s_2 = \lim s_3$.

(13) If s_2 is convergent and s_3 is convergent, then $\lim(s_2+s_3) = \lim s_2+\lim s_3$.

(14) If s_2 is convergent and s_3 is convergent, then $\lim(s_2-s_3) = \lim s_2-\lim s_3$.

(15) If s_1 is convergent, then $\lim(z \cdot s_1) = z \cdot \lim s_1$.

(16) If s_1 is convergent, then $\lim(-s_1) = -\lim s_1$.

(17) If s_1 is convergent, then $\lim(s_1 + x) = \lim s_1 + x$.

(18) If s_1 is convergent, then $\lim(s_1 - x) = \lim s_1 - x$.

(19) Suppose s_1 is convergent. Then $\lim s_1 = g$ if and only if for every r such that $r > 0$ there exists m such that for every n such that $n \geqslant m$ holds $\|s_1(n) - g\| < r$.

Let us consider X, s_1. The functor $\|s_1\|$ yielding a sequence of real numbers is defined as follows:

(Def. 3) For every n holds $\|s_1\|(n) = \|s_1(n)\|$.

One can prove the following three propositions:

(20) If s_1 is convergent, then $\|s_1\|$ is convergent.

(21) If s_1 is convergent and $\lim s_1 = g$, then $\|s_1\|$ is convergent and $\lim\|s_1\| = \|g\|$.

(22) If s_1 is convergent and $\lim s_1 = g$, then $\|s_1 - g\|$ is convergent and $\lim\|s_1 - g\| = 0$.

Let us consider X, s_1, x. The functor $\rho(s_1, x)$ yielding a sequence of real numbers is defined as follows:

(Def. 4) For every n holds $(\rho(s_1, x))(n) = \rho(s_1(n), x)$.

One can prove the following propositions:

(23) If s_1 is convergent and $\lim s_1 = g$, then $\rho(s_1, g)$ is convergent.

(24) If s_1 is convergent and $\lim s_1 = g$, then $\rho(s_1, g)$ is convergent and $\lim \rho(s_1, g) = 0$.

(25) If s_2 is convergent and $\lim s_2 = g_1$ and s_3 is convergent and $\lim s_3 = g_2$, then $\|s_2 + s_3\|$ is convergent and $\lim\|s_2 + s_3\| = \|g_1 + g_2\|$.

(26) If s_2 is convergent and $\lim s_2 = g_1$ and s_3 is convergent and $\lim s_3 = g_2$, then $\|(s_2+s_3)-(g_1+g_2)\|$ is convergent and $\lim\|(s_2+s_3)-(g_1+g_2)\| = 0$.

(27) If s_2 is convergent and $\lim s_2 = g_1$ and s_3 is convergent and $\lim s_3 = g_2$, then $\|s_2 - s_3\|$ is convergent and $\lim\|s_2 - s_3\| = \|g_1 - g_2\|$.

(28) If s_2 is convergent and $\lim s_2 = g_1$ and s_3 is convergent and $\lim s_3 = g_2$, then $\|s_2 - s_3 - (g_1 - g_2)\|$ is convergent and $\lim\|s_2 - s_3 - (g_1 - g_2)\| = 0$.

(29) If s_1 is convergent and $\lim s_1 = g$, then $\|z \cdot s_1\|$ is convergent and $\lim\|z \cdot s_1\| = \|z \cdot g\|$.

(30) If s_1 is convergent and $\lim s_1 = g$, then $\|z \cdot s_1 - z \cdot g\|$ is convergent and $\lim\|z \cdot s_1 - z \cdot g\| = 0$.

(31) If s_1 is convergent and $\lim s_1 = g$, then $\|-s_1\|$ is convergent and $\lim\|-s_1\| = \|-g\|$.

(32) If s_1 is convergent and $\lim s_1 = g$, then $\|-s_1 - -g\|$ is convergent and $\lim\|-s_1 - -g\| = 0$.

(33) If s_1 is convergent and $\lim s_1 = g$, then $\|(s_1+x)-(g+x)\|$ is convergent and $\lim\|(s_1 + x) - (g + x)\| = 0$.

(34) If s_1 is convergent and $\lim s_1 = g$, then $\|s_1 - x\|$ is convergent and $\lim\|s_1 - x\| = \|g - x\|$.

(35) If s_1 is convergent and $\lim s_1 = g$, then $\|s_1 - x - (g - x)\|$ is convergent and $\lim\|s_1 - x - (g - x)\| = 0$.

(36) If s_2 is convergent and $\lim s_2 = g_1$ and s_3 is convergent and $\lim s_3 = g_2$, then $\rho(s_2 + s_3, g_1 + g_2)$ is convergent and $\lim \rho(s_2 + s_3, g_1 + g_2) = 0$.

(37) If s_2 is convergent and $\lim s_2 = g_1$ and s_3 is convergent and $\lim s_3 = g_2$, then $\rho(s_2 - s_3, g_1 - g_2)$ is convergent and $\lim \rho(s_2 - s_3, g_1 - g_2) = 0$.

(38) If s_1 is convergent and $\lim s_1 = g$, then $\rho(z \cdot s_1, z \cdot g)$ is convergent and $\lim \rho(z \cdot s_1, z \cdot g) = 0$.

(39) If s_1 is convergent and $\lim s_1 = g$, then $\rho(s_1+x, g+x)$ is convergent and $\lim \rho(s_1 + x, g + x) = 0$.

Let us consider X, x, r. The functor $\mathrm{Ball}(x, r)$ yields a subset of X and is defined by:

(Def. 5) $\mathrm{Ball}(x, r) = \{y; y \text{ ranges over points of } X\colon \|x - y\| < r\}$.

The functor $\overline{\mathrm{Ball}}(x, r)$ yielding a subset of X is defined by:

(Def. 6) $\overline{\mathrm{Ball}}(x, r) = \{y; y \text{ ranges over points of } X\colon \|x - y\| \leqslant r\}$.

The functor $\mathrm{Sphere}(x, r)$ yielding a subset of X is defined as follows:

(Def. 7) $\mathrm{Sphere}(x, r) = \{y; y \text{ ranges over points of } X\colon \|x - y\| = r\}$.

Next we state a number of propositions:

(40) $w \in \mathrm{Ball}(x, r)$ iff $\|x - w\| < r$.

(41) $w \in \mathrm{Ball}(x, r)$ iff $\rho(x, w) < r$.

(42) If $r > 0$, then $x \in \mathrm{Ball}(x, r)$.

(43) If $y \in \mathrm{Ball}(x, r)$ and $w \in \mathrm{Ball}(x, r)$, then $\rho(y, w) < 2 \cdot r$.

(44) If $y \in \mathrm{Ball}(x, r)$, then $y - w \in \mathrm{Ball}(x - w, r)$.

(45) If $y \in \mathrm{Ball}(x, r)$, then $y - x \in \mathrm{Ball}(0_X, r)$.

(46) If $y \in \mathrm{Ball}(x, r)$ and $r \leqslant q$, then $y \in \mathrm{Ball}(x, q)$.

(47) $w \in \overline{\mathrm{Ball}}(x, r)$ iff $\|x - w\| \leqslant r$.

(48) $w \in \overline{\mathrm{Ball}}(x, r)$ iff $\rho(x, w) \leqslant r$.

(49) If $r \geqslant 0$, then $x \in \overline{\mathrm{Ball}}(x, r)$.

(50) If $y \in \mathrm{Ball}(x, r)$, then $y \in \overline{\mathrm{Ball}}(x, r)$.

(51) $w \in \mathrm{Sphere}(x, r)$ iff $\|x - w\| = r$.

(52) $w \in \mathrm{Sphere}(x, r)$ iff $\rho(x, w) = r$.

(53) If $y \in \mathrm{Sphere}(x, r)$, then $y \in \overline{\mathrm{Ball}}(x, r)$.

(54) $\mathrm{Ball}(x, r) \subseteq \overline{\mathrm{Ball}}(x, r)$.

(55) $\mathrm{Sphere}(x, r) \subseteq \overline{\mathrm{Ball}}(x, r)$.

(56) $\mathrm{Ball}(x, r) \cup \mathrm{Sphere}(x, r) = \overline{\mathrm{Ball}}(x, r)$.

2. Cauchy Sequence and Hilbert Space with Complex Coefficient

Let us consider X and let us consider s_1. We say that s_1 is Cauchy if and only if:

(Def. 8) For every r such that $r > 0$ there exists k such that for all n, m such that $n \geqslant k$ and $m \geqslant k$ holds $\rho(s_1(n), s_1(m)) < r$.

The following propositions are true:

(57) If s_1 is constant, then s_1 is Cauchy.

(58) s_1 is Cauchy if and only if for every r such that $r > 0$ there exists k such that for all n, m such that $n \geqslant k$ and $m \geqslant k$ holds $\|s_1(n) - s_1(m)\| < r$.

(59) If s_2 is Cauchy and s_3 is Cauchy, then $s_2 + s_3$ is Cauchy.

(60) If s_2 is Cauchy and s_3 is Cauchy, then $s_2 - s_3$ is Cauchy.

(61) If s_1 is Cauchy, then $z \cdot s_1$ is Cauchy.

(62) If s_1 is Cauchy, then $-s_1$ is Cauchy.

(63) If s_1 is Cauchy, then $s_1 + x$ is Cauchy.

(64) If s_1 is Cauchy, then $s_1 - x$ is Cauchy.

(65) If s_1 is convergent, then s_1 is Cauchy.

Let us consider X and let us consider s_2, s_3. We say that s_2 is compared to s_3 if and only if:

(Def. 9) For every r such that $r > 0$ there exists m such that for every n such that $n \geqslant m$ holds $\rho(s_2(n), s_3(n)) < r$.

One can prove the following two propositions:

(66) s_1 is compared to s_1.

(67) If s_2 is compared to s_3, then s_3 is compared to s_2.

Let us consider X and let us consider s_2, s_3. Let us notice that the predicate s_2 is compared to s_3 is reflexive and symmetric.

The following propositions are true:

(68) If s_2 is compared to s_3 and s_3 is compared to s_4, then s_2 is compared to s_4.

(69) s_2 is compared to s_3 iff for every r such that $r > 0$ there exists m such that for every n such that $n \geqslant m$ holds $\|s_2(n) - s_3(n)\| < r$.

(70) If there exists k such that for every n such that $n \geqslant k$ holds $s_2(n) = s_3(n)$, then s_2 is compared to s_3.

(71) If s_2 is Cauchy and compared to s_3, then s_3 is Cauchy.

(72) If s_2 is convergent and compared to s_3, then s_3 is convergent.

(73) If s_2 is convergent and $\lim s_2 = g$ and s_2 is compared to s_3, then s_3 is convergent and $\lim s_3 = g$.

Let us consider X and let us consider s_1. We say that s_1 is bounded if and only if:

(Def. 10) There exists M such that $M > 0$ and for every n holds $\|s_1(n)\| \leqslant M$.

We now state several propositions:

(74) If s_2 is bounded and s_3 is bounded, then $s_2 + s_3$ is bounded.

(75) If s_1 is bounded, then $-s_1$ is bounded.

(76) If s_2 is bounded and s_3 is bounded, then $s_2 - s_3$ is bounded.

(77) If s_1 is bounded, then $z \cdot s_1$ is bounded.

(78) If s_1 is constant, then s_1 is bounded.

(79) For every m there exists M such that $M > 0$ and for every n such that $n \leqslant m$ holds $\|s_1(n)\| < M$.

(80) If s_1 is convergent, then s_1 is bounded.

(81) If s_2 is bounded and compared to s_3, then s_3 is bounded.

Let us consider X, N_1, s_1. Then $s_1 \cdot N_1$ is a sequence of X.

We now state several propositions:

(82) Let X be a complex unitary space, s be a sequence of X, N be an increasing sequence of naturals, and n be a natural number. Then $(s \cdot N)(n) = s(N(n))$.

(83) s_1 is a subsequence of s_1.

(84) If s_2 is a subsequence of s_3 and s_3 is a subsequence of s_4, then s_2 is a subsequence of s_4.

(85) If s_1 is constant and s_2 is a subsequence of s_1, then s_2 is constant.

(86) If s_1 is constant and s_2 is a subsequence of s_1, then $s_1 = s_2$.

(87) If s_1 is bounded and s_2 is a subsequence of s_1, then s_2 is bounded.

(88) If s_1 is convergent and s_2 is a subsequence of s_1, then s_2 is convergent.

(89) If s_2 is a subsequence of s_1 and s_1 is convergent, then $\lim s_2 = \lim s_1$.

(90) If s_1 is Cauchy and s_2 is a subsequence of s_1, then s_2 is Cauchy.

Let us consider X, let us consider s_1, and let us consider k. The functor $s_1 \uparrow k$ yields a sequence of X and is defined as follows:

(Def. 11) For every n holds $(s_1 \uparrow k)(n) = s_1(n + k)$.

One can prove the following propositions:

(91) $s_1 \uparrow 0 = s_1$.

(92) $s_1 \uparrow k \uparrow m = s_1 \uparrow m \uparrow k$.

(93) $s_1 \uparrow k \uparrow m = s_1 \uparrow (k + m)$.

(94) $(s_2 + s_3) \uparrow k = s_2 \uparrow k + s_3 \uparrow k$.

(95) $(-s_1) \uparrow k = -s_1 \uparrow k$.

(96) $(s_2 - s_3) \uparrow k = s_2 \uparrow k - s_3 \uparrow k$.

(97) $(z \cdot s_1) \uparrow k = z \cdot (s_1 \uparrow k)$.

(98) $(s_1 \cdot N_1) \uparrow k = s_1 \cdot (N_1 \uparrow k)$.

(99) $s_1 \uparrow k$ is a subsequence of s_1.

(100) If s_1 is convergent, then $s_1 \uparrow k$ is convergent and $\lim(s_1 \uparrow k) = \lim s_1$.

(101) If s_1 is convergent and there exists k such that $s_1 = s_2 \uparrow k$, then s_2 is convergent.

(102) If s_1 is Cauchy and there exists k such that $s_1 = s_2 \uparrow k$, then s_2 is Cauchy.

(103) If s_1 is Cauchy, then $s_1 \uparrow k$ is Cauchy.

(104) If s_2 is compared to s_3, then $s_2 \uparrow k$ is compared to $s_3 \uparrow k$.

(105) If s_1 is bounded, then $s_1 \uparrow k$ is bounded.

(106) If s_1 is constant, then $s_1 \uparrow k$ is constant.

Let us consider X. We say that X is complete if and only if:

(Def. 12) For every s_1 such that s_1 is Cauchy holds s_1 is convergent.

The following proposition is true

(107) If X is complete and s_1 is Cauchy, then s_1 is bounded.

Let us consider X. We say that X is Hilbert if and only if:

(Def. 13) X is a complex unitary space and complete.

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