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# Brouwer Fixed Point Theorem for Disks on the Plane

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**Summary.** The article formalizes the proof of Brouwer's Fixed Point Theorem for 2-dimensional disks. Assuming, on the contrary, that the theorem is false, we show that a circle is a retract of a disk. Next, using the retraction, we prove that any loop in the circle is homotopic to the constant loop what contradicts with infiniteness of the fundamental group of a circle, see [15].

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The terminology and notation used in this paper are introduced in the following papers: [26], [9], [29], [2], [22], [28], [30], [6], [8], [7], [5], [4], [12], [3], [25], [16], [23], [21], [20], [27], [11], [13], [14], [18], [17], [19], [10], [1], and [24].

In this paper $n$ is a natural number, $a$, $r$ are real numbers, and $x$ is a point of $\mathcal{E}^n_{\mathrm{T}}$.

Let $S$, $T$ be non empty topological spaces. The functor DiffElems$(S,T)$ yielding a subset of $[\!:S,\,T:\!]$ is defined by:

(Def. 1) DiffElems$(S,T) = \{\langle s,\,t\rangle;\, s$ ranges over points of $S$, $t$ ranges over points of $T$: $s \neq t\}$.

One can prove the following proposition

(1) Let $S$, $T$ be non empty topological spaces and $x$ be a set. Then $x \in$ DiffElems$(S,T)$ if and only if there exists a point $s$ of $S$ and there exists a point $t$ of $T$ such that $x = \langle s,\,t\rangle$ and $s \neq t$.

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Let $S$ be a non trivial non empty topological space and let $T$ be a non empty topological space. One can check that DiffElems$(S, T)$ is non empty.

Let $S$ be a non empty topological space and let $T$ be a non trivial non empty topological space. Note that DiffElems$(S, T)$ is non empty.

We now state the proposition

(2) $\overline{\text{Ball}}(x, 0) = \{x\}$.

Let $n$ be a natural number, let $x$ be a point of $\mathcal{E}^n_{\text{T}}$, and let $r$ be a real number. The functor Tdisk$(x, r)$ yields a subspace of $\mathcal{E}^n_{\text{T}}$ and is defined by:

(Def. 2) Tdisk$(x, r) = (\mathcal{E}^n_{\text{T}}) \restriction \overline{\text{Ball}}(x, r)$.

Let $n$ be a natural number, let $x$ be a point of $\mathcal{E}^n_{\text{T}}$, and let $r$ be a non negative real number. Note that Tdisk$(x, r)$ is non empty.

We now state the proposition

(3) The carrier of Tdisk$(x, r) = \overline{\text{Ball}}(x, r)$.

Let $n$ be a natural number, let $x$ be a point of $\mathcal{E}^n_{\text{T}}$, and let $r$ be a real number. Note that Tdisk$(x, r)$ is convex.

We adopt the following convention: $n$ denotes a natural number, $r$ denotes a non negative real number, and $s$, $t$, $x$ denote points of $\mathcal{E}^n_{\text{T}}$.

One can prove the following two propositions:

(4) If $s \neq t$ and $s$ is a point of Tdisk$(x, r)$ and $s$ is not a point of Tcircle$(x, r)$, then there exists a point $e$ of Tcircle$(x, r)$ such that $\{e\} = \text{halfline}(s, t) \cap \text{Sphere}(x, r)$.

(5) Suppose $s \neq t$ and $s \in$ the carrier of Tcircle$(x, r)$ and $t$ is a point of Tdisk$(x, r)$. Then there exists a point $e$ of Tcircle$(x, r)$ such that $e \neq s$ and $\{s, e\} = \text{halfline}(s, t) \cap \text{Sphere}(x, r)$.

Let $n$ be a non empty natural number, let $o$ be a point of $\mathcal{E}^n_{\text{T}}$, let $s$, $t$ be points of $\mathcal{E}^n_{\text{T}}$, and let $r$ be a non negative real number. Let us assume that $s$ is a point of Tdisk$(o, r)$, and $t$ is a point of Tdisk$(o, r)$ and $s \neq t$. The functor HC$(s, t, o, r)$ yields a point of $\mathcal{E}^n_{\text{T}}$ and is defined as follows:

(Def. 3) HC$(s, t, o, r) \in \text{halfline}(s, t) \cap \text{Sphere}(o, r)$ and HC$(s, t, o, r) \neq s$.

In the sequel $n$ is a non empty natural number and $s$, $t$, $o$ are points of $\mathcal{E}^n_{\text{T}}$.

We now state three propositions:

(6) If $s$ is a point of Tdisk$(o, r)$ and $t$ is a point of Tdisk$(o, r)$ and $s \neq t$, then HC$(s, t, o, r)$ is a point of Tcircle$(o, r)$.

(7) Let $S$, $T$, $O$ be elements of $\mathcal{R}^n$. Suppose $S = s$ and $T = t$ and $O = o$. Suppose $s$ is a point of Tdisk$(o, r)$ and $t$ is a point of Tdisk$(o, r)$ and $s \neq t$ and $a = \frac{-|(t-s, s-o)| + \sqrt{|(t-s, s-o)|^{\mathbf{2}} - \sum^2(T-S) \cdot (\sum^2(S-O) - r^{\mathbf{2}})}}{\sum^2(T-S)}$. Then HC$(s, t, o, r) = (1 - a) \cdot s + a \cdot t$.

(8) Let $r_1$, $r_2$, $s_1$, $s_2$ be real numbers and $s$, $t$, $o$ be points of $\mathcal{E}^2_{\text{T}}$. Suppose that $s$ is a point of Tdisk$(o, r)$ and $t$ is a point of Tdisk$(o, r)$ and

$s \neq t$ and $r_1 = t_{\mathbf{1}} - s_{\mathbf{1}}$ and $r_2 = t_{\mathbf{2}} - s_{\mathbf{2}}$ and $s_1 = s_{\mathbf{1}} - o_{\mathbf{1}}$ and $s_2 = s_{\mathbf{2}} - o_{\mathbf{2}}$ and $a = \frac{-(s_1 \cdot r_1 + s_2 \cdot r_2) + \sqrt{(s_1 \cdot r_1 + s_2 \cdot r_2)^{\mathbf{2}} - (r_1{}^{\mathbf{2}} + r_2{}^{\mathbf{2}}) \cdot ((s_1{}^{\mathbf{2}} + s_2{}^{\mathbf{2}}) - r^{\mathbf{2}})}}{r_1{}^{\mathbf{2}} + r_2{}^{\mathbf{2}}}$. Then $\mathrm{HC}(s, t, o, r) = [s_{\mathbf{1}} + a \cdot r_1, s_{\mathbf{2}} + a \cdot r_2]$.

Let $n$ be a non empty natural number, let $o$ be a point of $\mathcal{E}^n_{\mathrm{T}}$, let $r$ be a non negative real number, let $x$ be a point of $\mathrm{Tdisk}(o, r)$, and let $f$ be a map from $\mathrm{Tdisk}(o, r)$ into $\mathrm{Tdisk}(o, r)$. Let us assume that $x$ is not a fixpoint of $f$. The functor $\mathrm{HC}(x, f)$ yielding a point of $\mathrm{Tcircle}(o, r)$ is defined as follows:

(Def. 4) There exist points $y$, $z$ of $\mathcal{E}^n_{\mathrm{T}}$ such that $y = x$ and $z = f(x)$ and $\mathrm{HC}(x, f) = \mathrm{HC}(z, y, o, r)$.

The following two propositions are true:

(9) Let $x$ be a point of $\mathrm{Tdisk}(o, r)$ and $f$ be a map from $\mathrm{Tdisk}(o, r)$ into $\mathrm{Tdisk}(o, r)$. If $x$ is not a fixpoint of $f$ and $x$ is a point of $\mathrm{Tcircle}(o, r)$, then $\mathrm{HC}(x, f) = x$.

(10) Let $r$ be a positive real number, $o$ be a point of $\mathcal{E}^2_{\mathrm{T}}$, and $Y$ be a non empty subspace of $\mathrm{Tdisk}(o, r)$. If $Y = \mathrm{Tcircle}(o, r)$, then $Y$ is not a retract of $\mathrm{Tdisk}(o, r)$.

Let $n$ be a non empty natural number, let $r$ be a non negative real number, let $o$ be a point of $\mathcal{E}^n_{\mathrm{T}}$, and let $f$ be a map from $\mathrm{Tdisk}(o, r)$ into $\mathrm{Tdisk}(o, r)$. The functor BR-map $f$ yielding a map from $\mathrm{Tdisk}(o, r)$ into $\mathrm{Tcircle}(o, r)$ is defined as follows:

(Def. 5) For every point $x$ of $\mathrm{Tdisk}(o, r)$ holds $(\text{BR-map } f)(x) = \mathrm{HC}(x, f)$.

The following propositions are true:

(11) Let $o$ be a point of $\mathcal{E}^n_{\mathrm{T}}$, $x$ be a point of $\mathrm{Tdisk}(o, r)$, and $f$ be a map from $\mathrm{Tdisk}(o, r)$ into $\mathrm{Tdisk}(o, r)$. If $x$ is not a fixpoint of $f$ and $x$ is a point of $\mathrm{Tcircle}(o, r)$, then $(\text{BR-map } f)(x) = x$.

(12) For every continuous map $f$ from $\mathrm{Tdisk}(o, r)$ into $\mathrm{Tdisk}(o, r)$ such that $f$ has no fixpoint holds $\text{BR-map } f \restriction \mathrm{Sphere}(o, r) = \mathrm{id}_{\mathrm{Tcircle}(o,r)}$.

(13) Let $r$ be a positive real number, $o$ be a point of $\mathcal{E}^2_{\mathrm{T}}$, and $f$ be a continuous map from $\mathrm{Tdisk}(o, r)$ into $\mathrm{Tdisk}(o, r)$. If $f$ has no fixpoint, then BR-map $f$ is continuous.

(14) For every non negative real number $r$ and for every point $o$ of $\mathcal{E}^2_{\mathrm{T}}$ holds every continuous map from $\mathrm{Tdisk}(o, r)$ into $\mathrm{Tdisk}(o, r)$ has a fixpoint.

(15) Let $r$ be a non negative real number, $o$ be a point of $\mathcal{E}^2_{\mathrm{T}}$, and $f$ be a continuous map from $\mathrm{Tdisk}(o, r)$ into $\mathrm{Tdisk}(o, r)$. Then there exists a point $x$ of $\mathrm{Tdisk}(o, r)$ such that $f(x) = x$.

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