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Solving Roots of Polynomial Equation of Degree 2 and 3 with Complex Coefficients

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Summary. In the article, solving complex roots of polynomial equation of degree 2 and 3 with real coefficients and complex roots of polynomial equation of degree 2 and 3 with complex coefficients is discussed.

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The terminology and notation used here are introduced in the following articles: [20], [15], [2], [5], [3], [8], [17], [16], [14], [10], [12], [7], [18], [1], [13], [21], [9], [19], [11], [6], and [4].

1. Solving Complex Roots of Polynomial Equation of Degree 2 and 3 with Real Coefficients

We follow the rules: a, b, c, d, a', b', c', d', x, y, x_1, u, v are real numbers and s, t, h, z, z_1, z_2, z_3, z_4, s_1, s_2, s_3, p, q are elements of $\mathbb{C}$.

Let a be a real number and let us consider z. Then $a \cdot z$ is an element of $\mathbb{C}$ and it can be characterized by the condition:

(Def. 1) $a \cdot z = (a + 0i) \cdot z$.

Then $a + z$ is an element of $\mathbb{C}$ and it can be characterized by the condition:

(Def. 2) $a + z = z + (a + 0i)$.

Let us consider z. Then $z^{\mathbf{2}}$ is an element of $\mathbb{C}$ and it can be characterized by the condition:

(Def. 3) $z^{\mathbf{2}} = (\Re(z)^{\mathbf{2}} - \Im(z)^{\mathbf{2}}) + (2 \cdot (\Re(z) \cdot \Im(z)))i$.

Let us consider a, b, c, z. Then Poly2(a, b, c, z) is an element of $\mathbb{C}$.

The following propositions are true:

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(1) $(a + ci) \cdot (b + di) = (a \cdot b - c \cdot d) + (a \cdot d + b \cdot c)i$.

(2) If $z = x + yi$, then $z^{\mathbf{2}} = (x^{\mathbf{2}} - y^{\mathbf{2}}) + (2 \cdot x \cdot y)i$.

(3) For all a, b holds $(a + 0i) \cdot (b + 0i) = a \cdot b + 0i$.

(4) If $a \neq 0$ and $\Delta(a, b, c) \geqslant 0$ and $\mathrm{Poly2}(a, b, c, z) = 0$, then $z = \frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a}$ or $z = \frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a}$ or $z = -\frac{b}{2 \cdot a}$.

(5) If $a \neq 0$ and $\Delta(a, b, c) < 0$ and $\mathrm{Poly2}(a, b, c, z) = 0_{\mathbb{C}}$, then $z = -\frac{b}{2 \cdot a} + \frac{\sqrt{-\Delta(a,b,c)}}{2 \cdot a} i$ or $z = -\frac{b}{2 \cdot a} + (-\frac{\sqrt{-\Delta(a,b,c)}}{2 \cdot a})i$.

(6) If $b \neq 0$ and for every z holds $\mathrm{Poly2}(0, b, c, z) = 0_{\mathbb{C}}$, then $z = -\frac{c}{b}$.

(7) Let a, b, c be real numbers and z, z_1, z_2 be elements of $\mathbb{C}$. Suppose $a \neq 0$. Suppose that for every element z of $\mathbb{C}$ holds $\mathrm{Poly2}(a, b, c, z) = \mathrm{Quard}(a, z_1, z_2, z)$. Then $\frac{b}{a} + 0i = -(z_1 + z_2)$ and $\frac{c}{a} + 0i = z_1 \cdot z_2$.

Let z be an element of $\mathbb{C}$. The functor $z^{\mathbf{3}}$ yielding an element of $\mathbb{C}$ is defined by:

(Def. 4) $z^{\mathbf{3}} = z^{\mathbf{2}} \cdot z$.

Let a, b, c, d be real numbers and let z be an element of $\mathbb{C}$. The functor $\mathrm{Poly}_3(a, b, c, d, z)$ yielding an element of $\mathbb{C}$ is defined as follows:

(Def. 5) $\mathrm{Poly}_3(a, b, c, d, z) = a \cdot z^{\mathbf{3}} + b \cdot z^{\mathbf{2}} + c \cdot z + d$.

We now state a number of propositions:

(8) $(0_{\mathbb{C}})^{\mathbf{3}} = 0_{\mathbb{C}}$.

(9) $(1_{\mathbb{C}})^{\mathbf{3}} = 1_{\mathbb{C}}$.

(10) $(-1_{\mathbb{C}})^{\mathbf{3}} = -1_{\mathbb{C}}$.

(11) $\Re(z^{\mathbf{3}}) = \Re(z)^3 - 3 \cdot \Re(z) \cdot \Im(z)^{\mathbf{2}}$ and $\Im(z^{\mathbf{3}}) = -\Im(z)^3 + 3 \cdot \Re(z)^{\mathbf{2}} \cdot \Im(z)$.

(12) If for every z holds $\mathrm{Poly}_3(a, b, c, d, z) = \mathrm{Poly}_3(a', b', c', d', z)$, then $a = a'$ and $b = b'$ and $c = c'$ and $d = d'$.

(13) $(z + h)^{\mathbf{3}} = z^{\mathbf{3}} + 3 \cdot h \cdot z^{\mathbf{2}} + 3 \cdot h^{\mathbf{2}} \cdot z + h^{\mathbf{3}}$.

(14) $(z \cdot h)^{\mathbf{3}} = z^{\mathbf{3}} \cdot h^{\mathbf{3}}$.

(15) If $b \neq 0$ and $\mathrm{Poly}_3(0, b, c, d, z) = 0_{\mathbb{C}}$ and $\Delta(b, c, d) \geqslant 0$, then $z = \frac{-c+\sqrt{\Delta(b,c,d)}}{2 \cdot b}$ or $z = \frac{-c-\sqrt{\Delta(b,c,d)}}{2 \cdot b}$ or $z = -\frac{c}{2 \cdot b}$.

(16) If $b \neq 0$ and $\mathrm{Poly}_3(0, b, c, d, z) = 0_{\mathbb{C}}$ and $\Delta(b, c, d) < 0$, then $z = -\frac{c}{2 \cdot b} + \frac{\sqrt{-\Delta(b,c,d)}}{2 \cdot b} i$ or $z = -\frac{c}{2 \cdot b} + (-\frac{\sqrt{-\Delta(b,c,d)}}{2 \cdot b})i$.

(17) If $a \neq 0$ and $\mathrm{Poly}_3(a, 0, c, 0, z) = 0$ and $4 \cdot a \cdot c \leqslant 0$, then $z = \frac{\sqrt{-4 \cdot a \cdot c}}{2 \cdot a}$ or $z = \frac{-\sqrt{-4 \cdot a \cdot c}}{2 \cdot a}$ or $z = 0$.

(18) If $a \neq 0$ and $\mathrm{Poly}_3(a, b, c, 0, z) = 0$ and $\Delta(a, b, c) \geqslant 0$, then $z = \frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a}$ or $z = \frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a}$ or $z = -\frac{b}{2 \cdot a}$ or $z = 0$.

(19) If $a \neq 0$ and $\mathrm{Poly}_3(a, b, c, 0, z) = 0_{\mathbb{C}}$ and $\Delta(a, b, c) < 0$, then $z = -\frac{b}{2 \cdot a} + \frac{\sqrt{-\Delta(a,b,c)}}{2 \cdot a} i$ or $z = -\frac{b}{2 \cdot a} + (-\frac{\sqrt{-\Delta(a,b,c)}}{2 \cdot a})i$ or $z = 0_{\mathbb{C}}$.

(20) If $a \geqslant 0$ and $y^{\mathbf{2}} = a$, then $y = \sqrt{a}$ or $y = -\sqrt{a}$.

(21) Suppose $a \neq 0$ and $\mathrm{Poly}_3(a, 0, c, d, z) = 0_{\mathbb{C}}$ and $\Im(z) = 0$. Let given u, v. Suppose $\Re(z) = u + v$ and $3 \cdot v \cdot u + \frac{c}{a} = 0$. Then

(i) $z = \sqrt[3]{-\frac{d}{2 \cdot a} + \sqrt{\frac{d^{\mathbf{2}}}{4 \cdot a^{\mathbf{2}}} + (\frac{c}{3 \cdot a})^3}} + \sqrt[3]{-\frac{d}{2 \cdot a} - \sqrt{\frac{d^{\mathbf{2}}}{4 \cdot a^{\mathbf{2}}} + (\frac{c}{3 \cdot a})^3}}$, or

(ii) $z = \sqrt[3]{-\frac{d}{2 \cdot a} + \sqrt{\frac{d^{\mathbf{2}}}{4 \cdot a^{\mathbf{2}}} + (\frac{c}{3 \cdot a})^3}} + \sqrt[3]{-\frac{d}{2 \cdot a} + \sqrt{\frac{d^{\mathbf{2}}}{4 \cdot a^{\mathbf{2}}} + (\frac{c}{3 \cdot a})^3}}$, or

(iii) $z = \sqrt[3]{-\frac{d}{2 \cdot a} - \sqrt{\frac{d^{\mathbf{2}}}{4 \cdot a^{\mathbf{2}}} + (\frac{c}{3 \cdot a})^3}} + \sqrt[3]{-\frac{d}{2 \cdot a} - \sqrt{\frac{d^{\mathbf{2}}}{4 \cdot a^{\mathbf{2}}} + (\frac{c}{3 \cdot a})^3}}$.

(22) Suppose $a \neq 0$ and $\mathrm{Poly}_3(a, 0, c, d, z) = 0_{\mathbb{C}}$ and $\Im(z) \neq 0$. Let given u, v. Suppose $\Re(z) = u + v$ and $3 \cdot v \cdot u + \frac{c}{4 \cdot a} = 0$ and $\frac{c}{a} \geqslant 0$. Then

(i) $z = (\sqrt[3]{\frac{d}{16 \cdot a} + \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3}} + \sqrt[3]{\frac{d}{16 \cdot a} - \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3}}) + \sqrt{3 \cdot (\sqrt[3]{\frac{d}{16 \cdot a} + \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3}} + \sqrt[3]{\frac{d}{16 \cdot a} - \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3}})^{\mathbf{2}} + \frac{c}{a}}\, i$, or

(ii) $z = (\sqrt[3]{\frac{d}{16 \cdot a} + \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3}} + \sqrt[3]{\frac{d}{16 \cdot a} - \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3}}) + (-\sqrt{3 \cdot (\sqrt[3]{\frac{d}{16 \cdot a} + \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3}} + \sqrt[3]{\frac{d}{16 \cdot a} - \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3}})^{\mathbf{2}} + \frac{c}{a}})\, i$, or

(iii) $z = 2 \cdot \sqrt[3]{\frac{d}{16 \cdot a} + \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3}} + \sqrt{3 \cdot (2 \cdot \sqrt[3]{\frac{d}{16 \cdot a} + \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3}})^{\mathbf{2}} + \frac{c}{a}}\, i$, or

(iv) $z = 2 \cdot \sqrt[3]{\frac{d}{16 \cdot a} + \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3}} + (-\sqrt{3 \cdot (2 \cdot \sqrt[3]{\frac{d}{16 \cdot a} + \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3}})^{\mathbf{2}} + \frac{c}{a}})\, i$, or

(v) $z = 2 \cdot \sqrt[3]{\frac{d}{16 \cdot a} - \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3}} + \sqrt{3 \cdot (2 \cdot \sqrt[3]{\frac{d}{16 \cdot a} - \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3}})^{\mathbf{2}} + \frac{c}{a}}\, i$, or

(vi) $z = 2 \cdot \sqrt[3]{\frac{d}{16 \cdot a} - \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3}} + (-\sqrt{3 \cdot (2 \cdot \sqrt[3]{\frac{d}{16 \cdot a} - \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3}})^{\mathbf{2}} + \frac{c}{a}})\, i$.

(23) Suppose $a \neq 0$ and $\mathrm{Poly}_3(a, b, c, d, z) = 0_{\mathbb{C}}$ and $\Im(z) = 0$. Let given u, v, x_1. Suppose $x_1 = \Re(z) + \frac{b}{3 \cdot a}$ and $\Re(z) = (u + v) - \frac{b}{3 \cdot a}$ and $3 \cdot u \cdot v + \frac{3 \cdot a \cdot c - b^{\mathbf{2}}}{3 \cdot a^{\mathbf{2}}} = 0$. Then

(i) $z = ((\sqrt[3]{(-(\frac{b}{3\cdot a})^3 - \frac{3\cdot a\cdot d-b\cdot c}{6\cdot a^{\mathbf{2}}}) + \sqrt{\frac{(2\cdot(\frac{b}{3\cdot a})^3+\frac{3\cdot a\cdot d-b\cdot c}{3\cdot a^{\mathbf{2}}})^{\mathbf{2}}}{4} + (\frac{3\cdot a\cdot c-b^{\mathbf{2}}}{9\cdot a^{\mathbf{2}}})^3}} + \sqrt[3]{-(\frac{b}{3\cdot a})^3 - \frac{3\cdot a\cdot d-b\cdot c}{6\cdot a^{\mathbf{2}}} - \sqrt{\frac{(2\cdot(\frac{b}{3\cdot a})^3+\frac{3\cdot a\cdot d-b\cdot c}{3\cdot a^{\mathbf{2}}})^{\mathbf{2}}}{4} + (\frac{3\cdot a\cdot c-b^{\mathbf{2}}}{9\cdot a^{\mathbf{2}}})^3}}) - \frac{b}{3\cdot a}) + 0i$, or

(ii) $z = ((\sqrt[3]{(-(\frac{b}{3\cdot a})^3 - \frac{3\cdot a\cdot d-b\cdot c}{6\cdot a^{\mathbf{2}}}) + \sqrt{\frac{(2\cdot(\frac{b}{3\cdot a})^3+\frac{3\cdot a\cdot d-b\cdot c}{3\cdot a^{\mathbf{2}}})^{\mathbf{2}}}{4} + (\frac{3\cdot a\cdot c-b^{\mathbf{2}}}{9\cdot a^{\mathbf{2}}})^3}} + \sqrt[3]{(-(\frac{b}{3\cdot a})^3 - \frac{3\cdot a\cdot d-b\cdot c}{6\cdot a^{\mathbf{2}}}) + \sqrt{\frac{(2\cdot(\frac{b}{3\cdot a})^3+\frac{3\cdot a\cdot d-b\cdot c}{3\cdot a^{\mathbf{2}}})^{\mathbf{2}}}{4} + (\frac{3\cdot a\cdot c-b^{\mathbf{2}}}{9\cdot a^{\mathbf{2}}})^3}}) - \frac{b}{3\cdot a}) + 0i$, or

(iii) $z = ((\sqrt[3]{-(\frac{b}{3\cdot a})^3 - \frac{3\cdot a\cdot d-b\cdot c}{6\cdot a^{\mathbf{2}}} - \sqrt{\frac{(2\cdot(\frac{b}{3\cdot a})^3+\frac{3\cdot a\cdot d-b\cdot c}{3\cdot a^{\mathbf{2}}})^{\mathbf{2}}}{4} + (\frac{3\cdot a\cdot c-b^{\mathbf{2}}}{9\cdot a^{\mathbf{2}}})^3}} + \sqrt[3]{-(\frac{b}{3\cdot a})^3 - \frac{3\cdot a\cdot d-b\cdot c}{6\cdot a^{\mathbf{2}}} - \sqrt{\frac{(2\cdot(\frac{b}{3\cdot a})^3+\frac{3\cdot a\cdot d-b\cdot c}{3\cdot a^{\mathbf{2}}})^{\mathbf{2}}}{4} + (\frac{3\cdot a\cdot c-b^{\mathbf{2}}}{9\cdot a^{\mathbf{2}}})^3}}) - \frac{b}{3\cdot a}) + 0i$.

(24) If $z_1 \neq 0$ and $\mathrm{Poly1}(z_1, z_2, z) = 0$, then $z = -\frac{z_2}{z_1}$.

(25) If $z_2 \neq 0$, then it is not true that there exists z such that $\mathrm{Poly1}(0, z_2, z) = 0$.

2. Complex Roots of Polynomial Equation of Degree 2 and 3 with Complex Coefficients

Let us consider z_1, z_2, z_3, z. The functor $\mathrm{CPoly2}(z_1, z_2, z_3, z)$ yields an element of $\mathbb{C}$ and is defined by:

(Def. 6) $\mathrm{CPoly2}(z_1, z_2, z_3, z) = z_1 \cdot z^{\mathbf{2}} + z_2 \cdot z + z_3$.

We now state a number of propositions:

(26) If for every z holds $\mathrm{CPoly2}(z_1, z_2, z_3, z) = \mathrm{CPoly2}(s_1, s_2, s_3, z)$, then $z_1 = s_1$ and $z_2 = s_2$ and $z_3 = s_3$.

(27) $\frac{-a+\sqrt{a^{\mathbf{2}}+b^{\mathbf{2}}}}{2} \geqslant 0$ and $\frac{a+\sqrt{a^{\mathbf{2}}+b^{\mathbf{2}}}}{2} \geqslant 0$.

(28) If $z^{\mathbf{2}} = s$ and $\Im(s) \geqslant 0$, then $z = \sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + \sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}}i$ or $z = -\sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + (-\sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}})i$.

(29) If $z^{\mathbf{2}} = s$ and $\Im(s) = 0$ and $\Re(s) > 0$, then $z = \sqrt{\Re(s)}$ or $z = -\sqrt{\Re(s)}$.

(30) If $z^{\mathbf{2}} = s$ and $\Im(s) = 0$ and $\Re(s) < 0$, then $z = 0 + \sqrt{-\Re(s)}i$ or $z = 0 + (-\sqrt{-\Re(s)})i$.

(31) If $z^{\mathbf{2}} = s$ and $\Im(s) < 0$, then $z = \sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + (-\sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}})i$ or $z = -\sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + \sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}}i$.

(32) Suppose $z^{\mathbf{2}} = s$. Then

(i) $z = \sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + \sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}}i$, or

(ii) $z = -\sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + (-\sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}})i$, or

(iii) $z = \sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + (-\sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}})i$, or

(iv) $z = -\sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + \sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}}i$.

(33) $\mathrm{CPoly2}(0_{\mathbb{C}}, 0_{\mathbb{C}}, 0_{\mathbb{C}}, z) = 0$.

(34) If $z_1 \neq 0$ and $\mathrm{CPoly2}(z_1, 0_{\mathbb{C}}, 0_{\mathbb{C}}, z) = 0$, then $z = 0$.

(35) If $z_1 \neq 0$ and $\mathrm{CPoly2}(z_1, z_2, 0_{\mathbb{C}}, z) = 0$, then $z = -\frac{z_2}{z_1}$ or $z = 0$.

(36) Suppose $z_1 \neq 0_{\mathbb{C}}$ and $\mathrm{CPoly2}(z_1, 0_{\mathbb{C}}, z_3, z) = 0_{\mathbb{C}}$. Let given s. Suppose $s = -\frac{z_3}{z_1}$. Then

(i) $z = \sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + \sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}}i$, or

(ii) $z = -\sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + (-\sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}})i$, or

(iii) $z = \sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + (-\sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}})i$, or

(iv) $z = -\sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + \sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}}i$.

(37) Suppose $z_1 \neq 0$ and $\mathrm{CPoly2}(z_1, z_2, z_3, z) = 0_{\mathbb{C}}$. Let given h, t. Suppose $h = (\frac{z_2}{2 \cdot z_1})^{\mathbf{2}} - \frac{z_3}{z_1}$ and $t = \frac{z_2}{2 \cdot z_1}$. Then

(i) $z = (\sqrt{\frac{\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}} + \sqrt{\frac{-\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}}i) - t$, or

(ii) $z = (-\sqrt{\frac{\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}} + (-\sqrt{\frac{-\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}})i) - t$, or

(iii) $z = (\sqrt{\frac{\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}} + (-\sqrt{\frac{-\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}})i) - t$, or

(iv) $z = (-\sqrt{\frac{\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}} + \sqrt{\frac{-\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}}i) - t$.

Let us consider z_1, z_2, z_3, z_4, z. The functor $\mathrm{CPoly2}(z_1, z_2, z_3, z_4, z)$ yields an element of $\mathbb{C}$ and is defined as follows:

(Def. 7) $\mathrm{CPoly2}(z_1, z_2, z_3, z_4, z) = z_1 \cdot z^{\mathbf{3}} + z_2 \cdot z^{\mathbf{2}} + z_3 \cdot z + z_4$.

One can prove the following propositions:

(38) If $z^{\mathbf{2}} = 1$, then $z = 1$ or $z = -1$.

(39) $z^3_{\mathbb{N}} = z \cdot z \cdot z$ and $z^3_{\mathbb{N}} = z^{\mathbf{2}} \cdot z$ and $z^3_{\mathbb{N}} = z^{\mathbf{3}}$.

(40) If $z_1 \neq 0$ and $\mathrm{CPoly2}(z_1, z_2, 0_{\mathbb{C}}, 0_{\mathbb{C}}, z) = 0_{\mathbb{C}}$, then $z = -\frac{z_2}{z_1}$ or $z = 0$.

(41) Suppose $z_1 \neq 0_{\mathbb{C}}$ and $\mathrm{CPoly2}(z_1, 0_{\mathbb{C}}, z_3, 0_{\mathbb{C}}, z) = 0_{\mathbb{C}}$. Let given s. Suppose $s = -\frac{z_3}{z_1}$. Then

(i) $z = 0_{\mathbb{C}}$, or

(ii) $z = \sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + \sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}}i$, or

(iii) $z = -\sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + (-\sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}})i$, or

(iv) $z = \sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + (-\sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}})i$, or

(v) $z = -\sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + \sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}}i$.

(42) Suppose $z_1 \neq 0$ and $\mathrm{CPoly2}(z_1, z_2, z_3, 0_{\mathbb{C}}, z) = 0_{\mathbb{C}}$. Let given s, h, t. Suppose $s = -\frac{z_3}{z_1}$ and $h = (\frac{z_2}{2\cdot z_1})^{\mathbf{2}} - \frac{z_3}{z_1}$ and $t = \frac{z_2}{2\cdot z_1}$. Then

(i) $z = 0$, or

(ii) $z = (\sqrt{\frac{\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}} + \sqrt{\frac{-\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}}i) - t$, or

(iii) $z = (-\sqrt{\frac{\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}} + (-\sqrt{\frac{-\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}})i) - t$, or

(iv) $z = (\sqrt{\frac{\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}} + (-\sqrt{\frac{-\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}})i) - t$, or

(v) $z = (-\sqrt{\frac{\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}} + \sqrt{\frac{-\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}}i) - t$.

(43) If $z = s - (\frac{1}{3} + 0i) \cdot z_1$, then $z^{\mathbf{2}} = s^{\mathbf{2}} + (-(\frac{2}{3} + 0i)) \cdot z_1 \cdot s + (\frac{1}{9} + 0i) \cdot {z_1}^{\mathbf{2}}$.

(44) If $z = s-(\frac{1}{3}+0i)\cdot z_1$, then $z^{\mathbf{3}} = ((s^{\mathbf{3}}-z_1\cdot s^{\mathbf{2}})+(\frac{1}{3}+0i)\cdot {z_1}^{\mathbf{2}}\cdot s)-(\frac{1}{27}+0i)\cdot {z_1}^{\mathbf{3}}$.

(45) Suppose $\mathrm{CPoly2}(1_{\mathbb{C}}, z_1, z_2, z_3, z) = 0_{\mathbb{C}}$. Let given p, q, s. Suppose $z = s - (\frac{1}{3} + 0i) \cdot z_1$ and $p = -(\frac{1}{3} + 0i) \cdot {z_1}^{\mathbf{2}} + z_2$ and $q = ((\frac{2}{27} + 0i) \cdot {z_1}^{\mathbf{3}} - (\frac{1}{3} + 0i) \cdot z_1 \cdot z_2) + z_3$. Then $\mathrm{CPoly2}(1_{\mathbb{C}}, 0_{\mathbb{C}}, p, q, s) = 0_{\mathbb{C}}$.

(46) For every element z of $\mathbb{C}$ holds $|z| \cdot \cos \operatorname{Arg} z + (|z| \cdot \sin \operatorname{Arg} z)i = (|z| + 0i) \cdot (\cos \operatorname{Arg} z + \sin \operatorname{Arg} zi)$.

(47) For every element z of $\mathbb{C}$ and for every natural number n holds $z_{\mathbb{N}}^{n+1} = (z_{\mathbb{N}}^{n}) \cdot z$.

(48) For every element z of $\mathbb{C}$ holds $z_{\mathbb{N}}^{1} = z$.

(49) For every element z of $\mathbb{C}$ holds $z_{\mathbb{N}}^{2} = z \cdot z$.

(50) For every natural number n such that $n > 0$ holds $0_{\mathbb{N}}^{n} = 0$.

(51) For all elements x, y of $\mathbb{C}$ and for every natural number n holds $(x \cdot y)_{\mathbb{N}}^{n} = (x_{\mathbb{N}}^{n}) \cdot y_{\mathbb{N}}^{n}$.

(52) For every real number x such that $x > 0$ and for every natural number n holds $(x + 0i)_{\mathbb{N}}^{n} = x^n + 0i$.

(53) For every real number x and for every natural number n holds $(\cos x + \sin xi)_{\mathbb{N}}^{n} = \cos(n \cdot x) + \sin(n \cdot x)i$.

(54) For every element z of $\mathbb{C}$ and for every natural number n such that $z \neq 0_{\mathbb{C}}$ or $n > 0$ holds $z_{\mathbb{N}}^{n} = |z|^n \cdot \cos(n \cdot \operatorname{Arg} z) + (|z|^n \cdot \sin(n \cdot \operatorname{Arg} z))i$.

(55) For all natural numbers n, k and for every real number x such that $n \neq 0$ holds $(\cos(\frac{x+2\cdot\pi\cdot k}{n}) + \sin(\frac{x+2\cdot\pi\cdot k}{n})i)_{\mathbb{N}}^{n} = \cos x + \sin xi$.

(56) Let z be an element of $\mathbb{C}$ and n, k be natural numbers. If $n \neq 0$, then $z = (\sqrt[n]{|z|} \cdot \cos(\frac{\operatorname{Arg} z+2\cdot\pi\cdot k}{n}) + (\sqrt[n]{|z|} \cdot \sin(\frac{\operatorname{Arg} z+2\cdot\pi\cdot k}{n}))i)_{\mathbb{N}}^{n}$.

Let z be an element of $\mathbb{C}$ and let n be a non empty natural number. An element of $\mathbb{C}$ is called a complex root of n, z if:

(Def. 8) $\mathrm{It}_{\mathbb{N}}^{n} = z$.

Next we state several propositions:

(57) Let z be an element of $\mathbb{C}$, n be a non empty natural number, and k be a natural number. Then $\sqrt[n]{|z|} \cdot \cos(\frac{\operatorname{Arg} z+2\cdot\pi\cdot k}{n}) + (\sqrt[n]{|z|} \cdot \sin(\frac{\operatorname{Arg} z+2\cdot\pi\cdot k}{n}))i$ is a complex root of n, z.

(58) For every element z of $\mathbb{C}$ and for every complex root v of 1, z holds $v = z$.

(59) For every non empty natural number n and for every complex root v of n, $0_{\mathbb{C}}$ holds $v = 0_{\mathbb{C}}$.

(60) Let n be a non empty natural number, z be an element of $\mathbb{C}$, and v be a complex root of n, z. If $v = 0_{\mathbb{C}}$, then $z = 0_{\mathbb{C}}$.

(61) Let n be a non empty natural number and k be a natural number. Then $\cos(\frac{2\cdot\pi\cdot k}{n}) + \sin(\frac{2\cdot\pi\cdot k}{n})i$ is a complex root of n, $1_{\mathbb{C}}$.

(62) For every natural number k holds $\cos(\frac{2\cdot\pi\cdot k}{3}) + \sin(\frac{2\cdot\pi\cdot k}{3})i$ is a complex root of 3, $1_{\mathbb{C}}$.

(63) For all elements z, s of $\mathbb{C}$ and for every natural number n such that $s \neq 0$ and $z \neq 0$ and $n \geqslant 1$ and $s_{\mathbb{N}}^n = z_{\mathbb{N}}^n$ holds $|s| = |z|$.

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