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Some Properties of Circles on the Plane[1]

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The articles [30], [34], [1], [5], [35], [7], [6], [23], [29], [17], [4], [33], [2], [27], [24], [26], [31], [9], [25], [37], [12], [18], [11], [10], [28], [3], [14], [36], [15], [32], [13], [16], [20], [19], [21], [8], and [22] provide the terminology and notation for this paper.

1. Preliminaries

For simplicity, we follow the rules: n is a natural number, i is an integer, a, b, r are real numbers, and x is a point of $\mathcal{E}^n_{\mathrm{T}}$.

One can check the following observations:

* $]0,1[$ is non empty,
* $[-1,1]$ is non empty, and
* $]\frac{1}{2},\frac{3}{2}[$ is non empty.

One can verify the following observations:

* the function sin is continuous,
* the function cos is continuous,
* the function arcsin is continuous, and
* the function arccos is continuous.

Next we state two propositions:

(1) $\sin(a\cdot r+b)=((\text{the function sin})\cdot\text{AffineMap}(a,b))(r)$.

(2) $\cos(a\cdot r+b)=((\text{the function cos})\cdot\text{AffineMap}(a,b))(r)$.

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Let a be a non zero real number and let b be a real number. Note that AffineMap(a, b) is onto and one-to-one.

Let a, b be real numbers. The functor IntIntervals(a, b) is defined as follows:

(Def. 1) IntIntervals$(a, b) = \{]a + n, b + n[: n$ ranges over elements of $\mathbb{Z}\}$.

One can prove the following proposition

(3) For every set x holds $x \in$ IntIntervals(a, b) iff there exists an element n of $\mathbb{Z}$ such that $x =]a + n, b + n[$.

Let a, b be real numbers. Observe that IntIntervals(a, b) is non empty.

Next we state the proposition

(4) If $b - a \leq 1$, then IntIntervals(a, b) is mutually-disjoint.

Let a, b be real numbers. Then IntIntervals(a, b) is a family of subsets of $\mathbb{R}^{\mathbf{1}}$.

Let a, b be real numbers. Then IntIntervals(a, b) is an open family of subsets of $\mathbb{R}^{\mathbf{1}}$.

2. Correspondence between $\mathbb{R}$ and $\mathbb{R}^{\mathbf{1}}$

Let r be a real number. The functor $R^1 r$ yielding a point of $\mathbb{R}^{\mathbf{1}}$ is defined by:

(Def. 2) $R^1 r = r$.

Let A be a subset of $\mathbb{R}$. The functor $R^1 A$ yielding a subset of $\mathbb{R}^{\mathbf{1}}$ is defined by:

(Def. 3) $R^1 A = A$.

Let A be a non empty subset of $\mathbb{R}$. Observe that $R^1 A$ is non empty.

Let A be an open subset of $\mathbb{R}$. Note that $R^1 A$ is open.

Let A be a closed subset of $\mathbb{R}$. Observe that $R^1 A$ is closed.

Let A be an open subset of $\mathbb{R}$. Observe that $\mathbb{R}^{\mathbf{1}} \restriction R^1 A$ is open.

Let A be a closed subset of $\mathbb{R}$. One can verify that $\mathbb{R}^{\mathbf{1}} \restriction R^1 A$ is closed.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. The functor $R^1 f$ yielding a map from $\mathbb{R}^{\mathbf{1}} \restriction R^1 \operatorname{dom} f$ into $\mathbb{R}^{\mathbf{1}} \restriction R^1 \operatorname{rng} f$ is defined as follows:

(Def. 4) $R^1 f = f$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. One can check that $R^1 f$ is onto.

Let f be an one-to-one partial function from $\mathbb{R}$ to $\mathbb{R}$. Observe that $R^1 f$ is one-to-one.

One can prove the following four propositions:

(5) $\mathbb{R}^{\mathbf{1}} \restriction R^1(\Omega_{\mathbb{R}}) = \mathbb{R}^{\mathbf{1}}$.

(6) For every partial function f from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} f = \mathbb{R}$ holds $\mathbb{R}^{\mathbf{1}} \restriction R^1 \operatorname{dom} f = \mathbb{R}^{\mathbf{1}}$.

(7) Every function f from $\mathbb{R}$ into $\mathbb{R}$ is a map from $\mathbb{R}^{\mathbf{1}}$ into $\mathbb{R}^{\mathbf{1}} \restriction R^1 \operatorname{rng} f$.

(8) Every function from $\mathbb{R}$ into $\mathbb{R}$ is a map from $\mathbb{R}^{\mathbf{1}}$ into $\mathbb{R}^{\mathbf{1}}$.

Let f be a continuous partial function from $\mathbb{R}$ to $\mathbb{R}$. Note that $R^1 f$ is continuous.

Let a be a non zero real number and let b be a real number. One can verify that $R^1\,\mathrm{AffineMap}(a, b)$ is open.

3. Circles

Let S be a subspace of $\mathcal{E}^2_{\mathrm{T}}$. We say that S satisfies conditions of simple closed curve if and only if:

(Def. 5) The carrier of S is a simple closed curve.

Let us note that every subspace of $\mathcal{E}^2_{\mathrm{T}}$ which satisfies conditions of simple closed curve is also non empty, arcwise connected, and compact.

Let r be a positive real number and let x be a point of $\mathcal{E}^2_{\mathrm{T}}$. Observe that $\mathrm{Sphere}(x, r)$ satisfies conditions of simple closed curve.

Let n be a natural number, let x be a point of $\mathcal{E}^n_{\mathrm{T}}$, and let r be a real number. The functor $\mathrm{Tcircle}(x, r)$ yielding a subspace of $\mathcal{E}^n_{\mathrm{T}}$ is defined by:

(Def. 6) $\mathrm{Tcircle}(x, r) = (\mathcal{E}^n_{\mathrm{T}}) \restriction \mathrm{Sphere}(x, r)$.

Let n be a non empty natural number, let x be a point of $\mathcal{E}^n_{\mathrm{T}}$, and let r be a non negative real number. Note that $\mathrm{Tcircle}(x, r)$ is strict and non empty.

One can prove the following proposition

(9) The carrier of $\mathrm{Tcircle}(x, r) = \mathrm{Sphere}(x, r)$.

Let n be a natural number, let x be a point of $\mathcal{E}^n_{\mathrm{T}}$, and let r be an empty real number. Note that $\mathrm{Tcircle}(x, r)$ is trivial.

Next we state the proposition

(10) $\mathrm{Tcircle}(0_{\mathcal{E}^2_{\mathrm{T}}}, r)$ is a subspace of $\mathrm{Trectangle}(-r, r, -r, r)$.

Let x be a point of $\mathcal{E}^2_{\mathrm{T}}$ and let r be a positive real number. One can verify that $\mathrm{Tcircle}(x, r)$ satisfies conditions of simple closed curve.

Let us mention that there exists a subspace of $\mathcal{E}^2_{\mathrm{T}}$ which is strict and satisfies conditions of simple closed curve.

Next we state the proposition

(11) For all subspaces S, T of $\mathcal{E}^2_{\mathrm{T}}$ satisfying conditions of simple closed curve holds S and T are homeomorphic.

Let n be a natural number. The functor $\mathrm{TopUnitCircle}\, n$ yields a subspace of $\mathcal{E}^n_{\mathrm{T}}$ and is defined by:

(Def. 7) $\mathrm{TopUnitCircle}\, n = \mathrm{Tcircle}(0_{\mathcal{E}^n_{\mathrm{T}}}, 1)$.

Let n be a non empty natural number. Note that $\mathrm{TopUnitCircle}\, n$ is non empty.

We now state several propositions:

(12) For every non empty natural number n and for every point x of $\mathcal{E}^n_{\mathrm{T}}$ such that x is a point of TopUnitCircle n holds $|x| = 1$.

(13) For every point x of $\mathcal{E}^2_{\mathrm{T}}$ such that x is a point of TopUnitCircle 2 holds $-1 \leq x_{\mathbf{1}}$ and $x_{\mathbf{1}} \leq 1$ and $-1 \leq x_{\mathbf{2}}$ and $x_{\mathbf{2}} \leq 1$.

(14) For every point x of $\mathcal{E}^2_{\mathrm{T}}$ such that x is a point of TopUnitCircle 2 and $x_{\mathbf{1}} = 1$ holds $x_{\mathbf{2}} = 0$.

(15) For every point x of $\mathcal{E}^2_{\mathrm{T}}$ such that x is a point of TopUnitCircle 2 and $x_{\mathbf{1}} = -1$ holds $x_{\mathbf{2}} = 0$.

(16) For every point x of $\mathcal{E}^2_{\mathrm{T}}$ such that x is a point of TopUnitCircle 2 and $x_{\mathbf{2}} = 1$ holds $x_{\mathbf{1}} = 0$.

(17) For every point x of $\mathcal{E}^2_{\mathrm{T}}$ such that x is a point of TopUnitCircle 2 and $x_{\mathbf{2}} = -1$ holds $x_{\mathbf{1}} = 0$.

The following propositions are true:

(18) TopUnitCircle 2 is a subspace of Trectangle$(-1, 1, -1, 1)$.

(19) Let n be a non empty natural number, r be a positive real number, x be a point of $\mathcal{E}^n_{\mathrm{T}}$, and f be a map from TopUnitCircle n into Tcircle(x, r). Suppose that for every point a of TopUnitCircle n and for every point b of $\mathcal{E}^n_{\mathrm{T}}$ such that $a = b$ holds $f(a) = r \cdot b + x$. Then f is a homeomorphism.

Let us observe that TopUnitCircle 2 satisfies conditions of simple closed curve.

One can prove the following proposition

(20) Let n be a non empty natural number, r, s be positive real numbers, and x, y be points of $\mathcal{E}^n_{\mathrm{T}}$. Then Tcircle$(x, r)$ and Tcircle(y, s) are homeomorphic.

Let x be a point of $\mathcal{E}^2_{\mathrm{T}}$ and let r be a non negative real number. Observe that Tcircle(x, r) is arcwise connected.

The point $c[10]$ of TopUnitCircle 2 is defined as follows:

(Def. 8) $c[10] = [1, 0]$.

The point $c[-10]$ of TopUnitCircle 2 is defined as follows:

(Def. 9) $c[-10] = [-1, 0]$.

Next we state the proposition

(21) $c[10] \neq c[-10]$.

Let p be a point of TopUnitCircle 2. The functor TopOpenUnitCircle p yielding a strict subspace of TopUnitCircle 2 is defined by:

(Def. 10) The carrier of TopOpenUnitCircle p = (the carrier of TopUnitCircle 2) $\setminus$ $\{p\}$.

Let p be a point of TopUnitCircle 2. Note that TopOpenUnitCircle p is non empty.

We now state several propositions:

(22) For every point p of TopUnitCircle 2 holds p is not a point of TopOpenUnitCircle p.

(23) For every point p of TopUnitCircle 2 holds TopOpenUnitCircle p = TopUnitCircle $2 \restriction (\Omega_{\text{TopUnitCircle } 2} \setminus \{p\})$.

(24) For all points p, q of TopUnitCircle 2 such that $p \neq q$ holds q is a point of TopOpenUnitCircle p.

(25) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that p is a point of TopOpenUnitCircle $c[10]$ and $p_{\mathbf{2}} = 0$ holds $p = c[-10]$.

(26) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that p is a point of TopOpenUnitCircle $c[-10]$ and $p_{\mathbf{2}} = 0$ holds $p = c[10]$.

Next we state three propositions:

(27) Let p be a point of TopUnitCircle 2 and x be a point of $\mathcal{E}^2_{\mathrm{T}}$. If x is a point of TopOpenUnitCircle p, then $-1 \leq x_{\mathbf{1}}$ and $x_{\mathbf{1}} \leq 1$ and $-1 \leq x_{\mathbf{2}}$ and $x_{\mathbf{2}} \leq 1$.

(28) For every point x of $\mathcal{E}^2_{\mathrm{T}}$ such that x is a point of TopOpenUnitCircle $c[10]$ holds $-1 \leq x_{\mathbf{1}}$ and $x_{\mathbf{1}} < 1$.

(29) For every point x of $\mathcal{E}^2_{\mathrm{T}}$ such that x is a point of TopOpenUnitCircle $c[-10]$ holds $-1 < x_{\mathbf{1}}$ and $x_{\mathbf{1}} \leq 1$.

Let p be a point of TopUnitCircle 2. Note that TopOpenUnitCircle p is open.

We now state two propositions:

(30) For every point p of TopUnitCircle 2 holds TopOpenUnitCircle p and $I(01)$ are homeomorphic.

(31) For all points p, q of TopUnitCircle 2 holds TopOpenUnitCircle p and TopOpenUnitCircle q are homeomorphic.

4. Correspondence between the Real Line and Circles

The map CircleMap from $\mathbb{R}^{\mathbf{1}}$ into TopUnitCircle 2 is defined by:

(Def. 11) For every real number x holds CircleMap$(x) = [\cos(2 \cdot \pi \cdot x), \sin(2 \cdot \pi \cdot x)]$.

Next we state several propositions:

(32) CircleMap(r) = CircleMap$(r + i)$.

(33) CircleMap$(i) = c[10]$.

(34) CircleMap$^{-1}(\{c[10]\}) = \mathbb{Z}$.

(35) If frac $r = \frac{1}{2}$, then CircleMap$(r) = [-1, 0]$.

(36) If frac $r = \frac{1}{4}$, then CircleMap$(r) = [0, 1]$.

(37) If frac $r = \frac{3}{4}$, then CircleMap$(r) = [0, -1]$.

(38) For all integers i, j holds CircleMap(r) = [((the function cos) $\cdot$ AffineMap$(2 \cdot \pi, 2 \cdot \pi \cdot i))(r)$, ((the function sin) $\cdot$ AffineMap$(2 \cdot \pi, 2 \cdot \pi \cdot j))(r)$].

Let us note that CircleMap is continuous.

The following proposition is true

(39) For every subset B of $\mathbb{R}^{\mathbf{1}}$ and for every map f from $\mathbb{R}^{\mathbf{1}}\restriction B$ into TopUnitCircle 2 such that $[0,1[\subseteq B$ and $f = \text{CircleMap}\restriction B$ holds f is onto.

Let us observe that CircleMap is onto.

Let r be a real number. One can verify that CircleMap $\restriction[r, r+1[$ is one-to-one.

Let r be a real number. One can verify that CircleMap $\restriction]r, r+1[$ is one-to-one.

One can prove the following two propositions:

(40) If $b - a \leq 1$, then for every set d such that $d \in \text{IntIntervals}(a,b)$ holds CircleMap $\restriction d$ is one-to-one.

(41) For every set d such that $d \in \text{IntIntervals}(a,b)$ holds $\text{CircleMap}^{\circ}\, d = \text{CircleMap}^{\circ} \bigcup \text{IntIntervals}(a,b)$.

Let r be a point of $\mathbb{R}^{\mathbf{1}}$. The functor CircleMap r yielding a map from $\mathbb{R}^{\mathbf{1}}\restriction R^1]r, r+1[$ into TopOpenUnitCircle CircleMap(r) is defined by:

(Def. 12) CircleMap r = CircleMap $\restriction]r, r+1[$.

One can prove the following proposition

(42) CircleMap $R^1(a+i) =$ CircleMap $R^1 a \cdot (\text{AffineMap}(1,-i)\restriction]a+i, a+i+1[)$.

Let r be a point of $\mathbb{R}^{\mathbf{1}}$. One can check that CircleMap r is one-to-one, onto, and continuous.

The map Circle2IntervalR from TopOpenUnitCircle $c[10]$ into $\mathbb{R}^{\mathbf{1}}\restriction R^1]0,1[$ is defined by the condition (Def. 13).

(Def. 13) Let p be a point of TopOpenUnitCircle $c[10]$. Then there exist real numbers x, y such that $p = [x,y]$ and if $y \geq 0$, then $\text{Circle2IntervalR}(p) = \frac{\arccos x}{2\cdot\pi}$ and if $y \leq 0$, then $\text{Circle2IntervalR}(p) = 1 - \frac{\arccos x}{2\cdot\pi}$.

The map Circle2IntervalL from TopOpenUnitCircle $c[-10]$ into $\mathbb{R}^{\mathbf{1}}\restriction R^1]\frac{1}{2}, \frac{3}{2}[$ is defined by the condition (Def. 14).

(Def. 14) Let p be a point of TopOpenUnitCircle $c[-10]$. Then there exist real numbers x, y such that $p = [x,y]$ and if $y \geq 0$, then $\text{Circle2IntervalL}(p) = 1 + \frac{\arccos x}{2\cdot\pi}$ and if $y \leq 0$, then $\text{Circle2IntervalL}(p) = 1 - \frac{\arccos x}{2\cdot\pi}$.

We now state two propositions:

(43) $(\text{CircleMap } R^1 0)^{-1} = \text{Circle2IntervalR}$.

(44) $(\text{CircleMap } R^1(\frac{1}{2}))^{-1} = \text{Circle2IntervalL}$.

Let us observe that Circle2IntervalR is one-to-one, onto, and continuous and Circle2IntervalL is one-to-one, onto, and continuous.

Let i be an integer. Observe that CircleMap $R^1 i$ is open and CircleMap $R^1(\frac{1}{2} + i)$ is open.

Let us observe that Circle2IntervalR is open and Circle2IntervalL is open.

Next we state several propositions:

(45) CircleMap $R^1 0$ is a homeomorphism.

(46) CircleMap $R^1(\frac{1}{2})$ is a homeomorphism.

(47) Circle2IntervalR is a homeomorphism.

(48) Circle2IntervalL is a homeomorphism.

(49) There exists a family F of subsets of TopUnitCircle 2 such that

(i) $F = \{\text{CircleMap}^{\circ}]0,1[, \text{CircleMap}^{\circ}]\frac{1}{2},\frac{3}{2}[\}$,

(ii) F is a cover of TopUnitCircle 2 and open, and

(iii) for every subset U of TopUnitCircle 2 holds if $U = \text{CircleMap}^{\circ}]0,1[$, then $\bigcup \text{IntIntervals}(0,1) = \text{CircleMap}^{-1}(U)$ and for every subset d of $\mathbb{R}^{\mathbf{1}}$ such that $d \in \text{IntIntervals}(0,1)$ and for every map f from $\mathbb{R}^{\mathbf{1}}\restriction d$ into TopUnitCircle $2\restriction U$ such that $f = \text{CircleMap}\restriction d$ holds f is a homeomorphism and if $U = \text{CircleMap}^{\circ}]\frac{1}{2},\frac{3}{2}[$, then $\bigcup \text{IntIntervals}(\frac{1}{2},\frac{3}{2}) = \text{CircleMap}^{-1}(U)$ and for every subset d of $\mathbb{R}^{\mathbf{1}}$ such that $d \in \text{IntIntervals}(\frac{1}{2},\frac{3}{2})$ and for every map f from $\mathbb{R}^{\mathbf{1}}\restriction d$ into TopUnitCircle $2\restriction U$ such that $f = \text{CircleMap}\restriction d$ holds f is a homeomorphism.

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