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On Some Points of a Simple Closed Curve[1]

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The notation and terminology used here are introduced in the following papers: [26], [28], [2], [13], [1], [29], [5], [18], [17], [3], [14], [24], [9], [23], [4], [25], [7], [10], [11], [12], [19], [20], [22], [21], [6], [8], [15], [16], and [27].

1. On the Subsets of $\mathcal{E}^2_{\mathrm{T}}$

For simplicity, we follow the rules: C denotes a simple closed curve, P denotes a subset of $\mathcal{E}^2_{\mathrm{T}}$, R denotes a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, p denotes a point of $\mathcal{E}^2_{\mathrm{T}}$, and i, j, k, m, n denote natural numbers.

One can prove the following propositions:

(1) For every point p of $\mathcal{E}^n_{\mathrm{T}}$ holds $\{p\}$ is Bounded.

(2) For all real numbers s_1, t and for every subset P of $\mathcal{E}^2_{\mathrm{T}}$ such that $P = \{[s, t]; s \text{ ranges over real numbers: } s_1 < s\}$ holds P is convex.

(3) For all real numbers s_2, t and for every subset P of $\mathcal{E}^2_{\mathrm{T}}$ such that $P = \{[s, t]; s \text{ ranges over real numbers: } s < s_2\}$ holds P is convex.

(4) For all real numbers s, t_1 and for every subset P of $\mathcal{E}^2_{\mathrm{T}}$ such that $P = \{[s, t]; t \text{ ranges over real numbers: } t_1 < t\}$ holds P is convex.

(5) For all real numbers s, t_2 and for every subset P of $\mathcal{E}^2_{\mathrm{T}}$ such that $P = \{[s, t]; t \text{ ranges over real numbers: } t < t_2\}$ holds P is convex.

(6) NorthHalfline $p \setminus \{p\}$ is convex.

(7) SouthHalfline $p \setminus \{p\}$ is convex.

(8) WestHalfline $p \setminus \{p\}$ is convex.

(9) EastHalfline $p \setminus \{p\}$ is convex.

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(10) For every subset A of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ holds UBD A misses A.

(11) Let P be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is an arc from p_1 to p_2 and $p_1 \neq q_1$ and $p_2 \neq q_2$. Then $p_1 \notin \mathrm{Segment}(P, p_1, p_2, q_1, q_2)$ and $p_2 \notin \mathrm{Segment}(P, p_1, p_2, q_1, q_2)$.

(12) $\mathrm{proj2}^\circ(C \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2}))$ is not empty.

(13) For every compact subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{proj2}^\circ(C \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2}))$ is closed, lower bounded, and upper bounded.

2. Gauges

The following propositions are true:

(14) $\langle 1,\ 1\rangle \in$ the indices of $\mathrm{Gauge}(R, n)$.

(15) $\langle 1,\ 2\rangle \in$ the indices of $\mathrm{Gauge}(R, n)$.

(16) $\langle 2,\ 1\rangle \in$ the indices of $\mathrm{Gauge}(R, n)$.

(17) Let C be a non vertical non horizontal compact subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $m > k$ and $\langle i,\ j\rangle \in$ the indices of $\mathrm{Gauge}(C, k)$ and $\langle i,\ j+1\rangle \in$ the indices of $\mathrm{Gauge}(C, k)$. Then $\rho(\mathrm{Gauge}(C, m) \circ (i, j), \mathrm{Gauge}(C, m) \circ (i, j+1)) < \rho(\mathrm{Gauge}(C, k) \circ (i, j), \mathrm{Gauge}(C, k) \circ (i, j+1))$.

(18) For every non vertical non horizontal compact subset C of $\mathcal{E}^2_{\mathrm{T}}$ such that $m > k$ holds $\rho(\mathrm{Gauge}(C, m) \circ (1, 1), \mathrm{Gauge}(C, m) \circ (1, 2)) < \rho(\mathrm{Gauge}(C, k) \circ (1, 1), \mathrm{Gauge}(C, k) \circ (1, 2))$.

(19) Let C be a non vertical non horizontal compact subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $m > k$ and $\langle i,\ j\rangle \in$ the indices of $\mathrm{Gauge}(C, k)$ and $\langle i+1,\ j\rangle \in$ the indices of $\mathrm{Gauge}(C, k)$. Then $\rho(\mathrm{Gauge}(C, m) \circ (i, j), \mathrm{Gauge}(C, m) \circ (i+1, j)) < \rho(\mathrm{Gauge}(C, k) \circ (i, j), \mathrm{Gauge}(C, k) \circ (i+1, j))$.

(20) For every non vertical non horizontal compact subset C of $\mathcal{E}^2_{\mathrm{T}}$ such that $m > k$ holds $\rho(\mathrm{Gauge}(C, m) \circ (1, 1), \mathrm{Gauge}(C, m) \circ (2, 1)) < \rho(\mathrm{Gauge}(C, k) \circ (1, 1), \mathrm{Gauge}(C, k) \circ (2, 1))$.

(21) Let r, t be real numbers. Suppose $r > 0$ and $t > 0$. Then there exists a natural number n such that $i < n$ and $\rho(\mathrm{Gauge}(C, n) \circ (1, 1), \mathrm{Gauge}(C, n) \circ (1, 2)) < r$ and $\rho(\mathrm{Gauge}(C, n) \circ (1, 1), \mathrm{Gauge}(C, n) \circ (2, 1)) < t$.

3. Middle Points

We now state four propositions:

(22) UpperMiddlePoint $C \in C$.

(23) LowerMiddlePoint $C \in C$.

(24) $(\mathrm{LowerMiddlePoint}\ C)_{\mathbf{2}} \neq (\mathrm{UpperMiddlePoint}\ C)_{\mathbf{2}}$.

(25) LowerMiddlePoint $C \neq$ UpperMiddlePoint C.

4. UpperArc and LowerArc

Next we state several propositions:

(26) W-bound(C) = W-bound(UpperArc(C)).

(27) E-bound(C) = E-bound(UpperArc(C)).

(28) W-bound(C) = W-bound(LowerArc(C)).

(29) E-bound(C) = E-bound(LowerArc(C)).

(30) UpperArc$(C) \cap$ VerticalLine$(\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2})$ is not empty and proj2°(UpperArc$(C) \cap$ VerticalLine$(\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2})$) is not empty.

(31) LowerArc$(C) \cap$ VerticalLine$(\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2})$ is not empty and proj2°(LowerArc$(C) \cap$ VerticalLine$(\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2})$) is not empty.

(32) For every compact connected subset P of $\mathcal{E}^2_{\mathrm{T}}$ such that $P \subseteq C$ and $\mathrm{W}_{\min}(C) \in P$ and $\mathrm{E}_{\max}(C) \in P$ holds UpperArc$(C) \subseteq P$ or LowerArc$(C) \subseteq P$.

5. UMP and LMP

Let P be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$. The functor UMP P yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 1) UMP $P = [\frac{\text{E-bound}(P)+\text{W-bound}(P)}{2}$, $\sup(\text{proj2}^\circ(P \cap \text{VerticalLine}(\frac{\text{E-bound}(P)+\text{W-bound}(P)}{2})))]$.

The functor LMP P yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 2) LMP $P = [\frac{\text{E-bound}(P)+\text{W-bound}(P)}{2}$, $\inf(\text{proj2}^\circ(P \cap \text{VerticalLine}(\frac{\text{E-bound}(P)+\text{W-bound}(P)}{2})))]$.

We now state a number of propositions:

(33) $(\text{UMP } P)_{\mathbf{1}} = \frac{\text{W-bound}(P)+\text{E-bound}(P)}{2}$.

(34) $(\text{UMP } P)_{\mathbf{2}} = \sup(\text{proj2}^\circ(P \cap \text{VerticalLine}(\frac{\text{E-bound}(P)+\text{W-bound}(P)}{2})))$.

(35) $(\text{LMP } P)_{\mathbf{1}} = \frac{\text{W-bound}(P)+\text{E-bound}(P)}{2}$.

(36) $(\text{LMP } P)_{\mathbf{2}} = \inf(\text{proj2}^\circ(P \cap \text{VerticalLine}(\frac{\text{E-bound}(P)+\text{W-bound}(P)}{2})))$.

(37) For every non vertical compact subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds UMP $C \neq \mathrm{W}_{\min}(C)$.

(38) For every non vertical compact subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds UMP $C \neq \mathrm{E}_{\max}(C)$.

(39) For every non vertical compact subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds LMP $C \neq \mathrm{W}_{\min}(C)$.

(40) For every non vertical compact subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds LMP $C \neq \mathrm{E}_{\max}(C)$.

(41) For every compact subset C of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in C \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2})$ holds $p_{\mathbf{2}} \leq (\mathrm{UMP}\, C)_{\mathbf{2}}$.

(42) For every compact subset C of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in C \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2})$ holds $(\mathrm{LMP}\, C)_{\mathbf{2}} \leq p_{\mathbf{2}}$.

(43) $\mathrm{UMP}\, C \in C$.

(44) $\mathrm{LMP}\, C \in C$.

(45) $\mathcal{L}(\mathrm{UMP}\, P, [\frac{\text{W-bound}(P)+\text{E-bound}(P)}{2}, \text{N-bound}(P)])$ is vertical.

(46) $\mathcal{L}(\mathrm{LMP}\, P, [\frac{\text{W-bound}(P)+\text{E-bound}(P)}{2}, \text{S-bound}(P)])$ is vertical.

(47) $\mathcal{L}(\mathrm{UMP}\, C, [\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2}, \text{N-bound}(C)]) \cap C = \{\mathrm{UMP}\, C\}$.

(48) $\mathcal{L}(\mathrm{LMP}\, C, [\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2}, \text{S-bound}(C)]) \cap C = \{\mathrm{LMP}\, C\}$.

(49) $(\mathrm{LMP}\, C)_{\mathbf{2}} < (\mathrm{UMP}\, C)_{\mathbf{2}}$.

(50) $\mathrm{UMP}\, C \neq \mathrm{LMP}\, C$.

(51) $\text{S-bound}(C) < (\mathrm{UMP}\, C)_{\mathbf{2}}$.

(52) $(\mathrm{UMP}\, C)_{\mathbf{2}} \leq \text{N-bound}(C)$.

(53) $\text{S-bound}(C) \leq (\mathrm{LMP}\, C)_{\mathbf{2}}$.

(54) $(\mathrm{LMP}\, C)_{\mathbf{2}} < \text{N-bound}(C)$.

(55) $\mathcal{L}(\mathrm{UMP}\, C, [\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2}, \text{N-bound}(C)])$ misses $\mathcal{L}(\mathrm{LMP}\, C, [\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2}, \text{S-bound}(C)])$.

(56) Let A, B be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $A \subseteq B$ and $\text{W-bound}(A) + \text{E-bound}(A) = \text{W-bound}(B) + \text{E-bound}(B)$ and $A \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(A)+\text{E-bound}(A)}{2})$ is non empty and $\mathrm{proj2}^\circ(B \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(A)+\text{E-bound}(A)}{2}))$ is upper bounded. Then $(\mathrm{UMP}\, A)_{\mathbf{2}} \leq (\mathrm{UMP}\, B)_{\mathbf{2}}$.

(57) Let A, B be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $A \subseteq B$ and $\text{W-bound}(A) + \text{E-bound}(A) = \text{W-bound}(B) + \text{E-bound}(B)$ and $A \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(A)+\text{E-bound}(A)}{2})$ is non empty and $\mathrm{proj2}^\circ(B \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(A)+\text{E-bound}(A)}{2}))$ is lower bounded. Then $(\mathrm{LMP}\, B)_{\mathbf{2}} \leq (\mathrm{LMP}\, A)_{\mathbf{2}}$.

(58) Let A, B be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $A \subseteq B$ and $\mathrm{UMP}\, B \in A$ and $A \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(A)+\text{E-bound}(A)}{2})$ is non empty and $\mathrm{proj2}^\circ(B \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(B)+\text{E-bound}(B)}{2}))$ is upper bounded and $\text{W-bound}(A) + \text{E-bound}(A) = \text{W-bound}(B) + \text{E-bound}(B)$. Then $\mathrm{UMP}\, A = \mathrm{UMP}\, B$.

(59) Let A, B be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $A \subseteq B$ and $\mathrm{LMP}\, B \in A$ and $A \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(A)+\text{E-bound}(A)}{2})$ is non empty and $\mathrm{proj2}^\circ(B \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(B)+\text{E-bound}(B)}{2}))$ is lower bounded and $\text{W-bound}(A) + \text{E-bound}(A) = \text{W-bound}(B) + \text{E-bound}(B)$. Then $\mathrm{LMP}\, A = \mathrm{LMP}\, B$.

(60) $(\mathrm{UMP}\ \mathrm{UpperArc}(C))_{\mathbf{2}} \leq \text{N-bound}(C)$.

(61) $\text{S-bound}(C) \leq (\mathrm{LMP}\ \mathrm{LowerArc}(C))_{\mathbf{2}}$.

(62) $\mathrm{LMP}\ C \notin \mathrm{LowerArc}(C)$ or $\mathrm{UMP}\ C \notin \mathrm{LowerArc}(C)$.

(63) $\mathrm{LMP}\ C \notin \mathrm{UpperArc}(C)$ or $\mathrm{UMP}\ C \notin \mathrm{UpperArc}(C)$.

(64) If $0 < n$, then $\sup(\mathrm{proj2}^\circ(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \cap \mathcal{L}(\mathrm{Gauge}(C,n) \circ (\mathrm{Center\ Gauge}(C,n),1), \mathrm{Gauge}(C,n) \circ (\mathrm{Center\ Gauge}(C,n), \mathrm{len\ Gauge}(C,n))))) = \sup(\mathrm{proj2}^\circ(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \cap \mathrm{VerticalLine}(\frac{\text{E-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) + \text{W-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))}{2}))))$.

(65) If $0 < n$, then $\inf(\mathrm{proj2}^\circ(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \cap \mathcal{L}(\mathrm{Gauge}(C,n) \circ (\mathrm{Center\ Gauge}(C,n),1), \mathrm{Gauge}(C,n) \circ (\mathrm{Center\ Gauge}(C,n), \mathrm{len\ Gauge}(C,n))))) = \inf(\mathrm{proj2}^\circ(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \cap \mathrm{VerticalLine}(\frac{\text{E-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) + \text{W-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))}{2}))))$.

(66) If $0 < n$, then $\mathrm{UMP}\ \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = [\frac{\text{E-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) + \text{W-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))}{2}, \sup(\mathrm{proj2}^\circ(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \cap \mathcal{L}(\mathrm{Gauge}(C,n) \circ (\mathrm{Center\ Gauge}(C,n),1), \mathrm{Gauge}(C,n) \circ (\mathrm{Center\ Gauge}(C,n), \mathrm{len\ Gauge}(C,n)))))]$.

(67) If $0 < n$, then $\mathrm{LMP}\ \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = [\frac{\text{E-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) + \text{W-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))}{2}, \inf(\mathrm{proj2}^\circ(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \cap \mathcal{L}(\mathrm{Gauge}(C,n) \circ (\mathrm{Center\ Gauge}(C,n),1), \mathrm{Gauge}(C,n) \circ (\mathrm{Center\ Gauge}(C,n), \mathrm{len\ Gauge}(C,n)))))]$.

(68) $(\mathrm{UMP}\ C)_{\mathbf{2}} < (\mathrm{UMP}\ \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))_{\mathbf{2}}$.

(69) $(\mathrm{LMP}\ C)_{\mathbf{2}} > (\mathrm{LMP}\ \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))_{\mathbf{2}}$.

(70) $\mathrm{UMP}\ \mathrm{UpperArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \in \mathrm{UpperArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$.

(71) $\mathrm{LMP}\ \mathrm{LowerArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \in \mathrm{LowerArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$.

(72) If $0 < n$, then there exists a natural number i such that $1 \leq i$ and $i \leq \mathrm{len\ Gauge}(C,n)$ and $\mathrm{UMP}\ \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = \mathrm{Gauge}(C,n) \circ (\mathrm{Center\ Gauge}(C,n), i)$.

(73) If $0 < n$, then there exists a natural number i such that $1 \leq i$ and $i \leq \mathrm{len\ Gauge}(C,n)$ and $\mathrm{LMP}\ \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = \mathrm{Gauge}(C,n) \circ (\mathrm{Center\ Gauge}(C,n), i)$.

(74) If $0 < n$, then $\mathrm{UMP}\ \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = \mathrm{UMP}\ \mathrm{UpperArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$.

(75) If $0 < n$, then $\mathrm{LMP}\ \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = \mathrm{LMP}\ \mathrm{LowerArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$.

(76) If $0 < n$, then $(\mathrm{UMP}\ C)_{\mathbf{2}} < (\mathrm{UMP}\ \mathrm{UpperArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))))_{\mathbf{2}}$.

(77) If $0 < n$, then $(\mathrm{LMP}\ \mathrm{LowerArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))))_{\mathbf{2}} < (\mathrm{LMP}\ C)_{\mathbf{2}}$.

(78) If $i \leq j$, then $(\mathrm{UMP}\ \widetilde{\mathcal{L}}(\mathrm{Cage}(C,j)))_{\mathbf{2}} \leq (\mathrm{UMP}\ \widetilde{\mathcal{L}}(\mathrm{Cage}(C,i)))_{\mathbf{2}}$.

(79) If $i \leq j$, then $(\mathrm{LMP}\ \widetilde{\mathcal{L}}(\mathrm{Cage}(C,i)))_{\mathbf{2}} \leq (\mathrm{LMP}\ \widetilde{\mathcal{L}}(\mathrm{Cage}(C,j)))_{\mathbf{2}}$.

(80) If $0 < i$ and $i \leq j$, then $(\mathrm{UMP}\ \mathrm{UpperArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,j))))_{\mathbf{2}} \leq (\mathrm{UMP}\ \mathrm{UpperArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,i))))_{\mathbf{2}}$.

(81) If $0 < i$ and $i \leq j$, then $(\text{LMP LowerArc}(\widetilde{\mathcal{L}}(\text{Cage}(C,i))))_{\mathbf{2}} \leq (\text{LMP LowerArc}(\widetilde{\mathcal{L}}(\text{Cage}(C,j))))_{\mathbf{2}}$.

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