

Inverse Trigonometric Functions Arcsin and Arccos¹

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Summary. Notions of inverse sine and inverse cosine have been introduced. Their basic properties have been proved.

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The papers [11], [14], [1], [10], [3], [13], [12], [9], [15], [2], [16], [6], [4], [5], [7], [8], and [17] provide the terminology and notation for this paper.

1. PRELIMINARIES

In this paper r, s are real numbers and i is an integer number.

We now state two propositions:

- (1) If $0 \leq r$ and $r < s$, then $\lfloor \frac{r}{s} \rfloor = 0$.
- (2) For every function f and for all sets X, Y such that $f|X$ is one-to-one and $Y \subseteq X$ holds $f|Y$ is one-to-one.

2. FUNCTIONS SINE AND COSINE

We now state four propositions:

- (3) $-1 \leq \sin r$.
- (4) $\sin r \leq 1$.

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$$(5) \quad -1 \leq \cos r.$$

$$(6) \quad \cos r \leq 1.$$

One can check that π is positive.

The following propositions are true:

$$(7) \quad \sin(-\frac{\pi}{2}) = -1 \text{ and (the function } \sin)(-\frac{\pi}{2}) = -1.$$

$$(8) \quad (\text{The function } \sin)(r) = (\text{the function } \sin)(r + 2 \cdot \pi \cdot i).$$

$$(9) \quad \cos(-\frac{\pi}{2}) = 0 \text{ and (the function } \cos)(-\frac{\pi}{2}) = 0.$$

$$(10) \quad (\text{The function } \cos)(r) = (\text{the function } \cos)(r + 2 \cdot \pi \cdot i).$$

$$(11) \quad \text{If } 2 \cdot \pi \cdot i < r \text{ and } r < \pi + 2 \cdot \pi \cdot i, \text{ then } \sin r > 0.$$

$$(12) \quad \text{If } \pi + 2 \cdot \pi \cdot i < r \text{ and } r < 2 \cdot \pi + 2 \cdot \pi \cdot i, \text{ then } \sin r < 0.$$

$$(13) \quad \text{If } -\frac{\pi}{2} + 2 \cdot \pi \cdot i < r \text{ and } r < \frac{\pi}{2} + 2 \cdot \pi \cdot i, \text{ then } \cos r > 0.$$

$$(14) \quad \text{If } \frac{\pi}{2} + 2 \cdot \pi \cdot i < r \text{ and } r < \frac{3}{2} \cdot \pi + 2 \cdot \pi \cdot i, \text{ then } \cos r < 0.$$

$$(15) \quad \text{If } \frac{3}{2} \cdot \pi + 2 \cdot \pi \cdot i < r \text{ and } r < 2 \cdot \pi + 2 \cdot \pi \cdot i, \text{ then } \cos r > 0.$$

$$(16) \quad \text{If } 2 \cdot \pi \cdot i \leq r \text{ and } r \leq \pi + 2 \cdot \pi \cdot i, \text{ then } \sin r \geq 0.$$

$$(17) \quad \text{If } \pi + 2 \cdot \pi \cdot i \leq r \text{ and } r \leq 2 \cdot \pi + 2 \cdot \pi \cdot i, \text{ then } \sin r \leq 0.$$

$$(18) \quad \text{If } -\frac{\pi}{2} + 2 \cdot \pi \cdot i \leq r \text{ and } r \leq \frac{\pi}{2} + 2 \cdot \pi \cdot i, \text{ then } \cos r \geq 0.$$

$$(19) \quad \text{If } \frac{\pi}{2} + 2 \cdot \pi \cdot i \leq r \text{ and } r \leq \frac{3}{2} \cdot \pi + 2 \cdot \pi \cdot i, \text{ then } \cos r \leq 0.$$

$$(20) \quad \text{If } \frac{3}{2} \cdot \pi + 2 \cdot \pi \cdot i \leq r \text{ and } r \leq 2 \cdot \pi + 2 \cdot \pi \cdot i, \text{ then } \cos r \geq 0.$$

$$(21) \quad \text{If } 2 \cdot \pi \cdot i \leq r \text{ and } r < 2 \cdot \pi + 2 \cdot \pi \cdot i \text{ and } \sin r = 0, \text{ then } r = 2 \cdot \pi \cdot i \text{ or } r = \pi + 2 \cdot \pi \cdot i.$$

$$(22) \quad \text{If } 2 \cdot \pi \cdot i \leq r \text{ and } r < 2 \cdot \pi + 2 \cdot \pi \cdot i \text{ and } \cos r = 0, \text{ then } r = \frac{\pi}{2} + 2 \cdot \pi \cdot i \text{ or } r = \frac{3}{2} \cdot \pi + 2 \cdot \pi \cdot i.$$

$$(23) \quad \text{If } \sin r = -1, \text{ then } r = \frac{3}{2} \cdot \pi + 2 \cdot \pi \cdot \lfloor \frac{r}{2 \cdot \pi} \rfloor.$$

$$(24) \quad \text{If } \sin r = 1, \text{ then } r = \frac{\pi}{2} + 2 \cdot \pi \cdot \lfloor \frac{r}{2 \cdot \pi} \rfloor.$$

$$(25) \quad \text{If } \cos r = -1, \text{ then } r = \pi + 2 \cdot \pi \cdot \lfloor \frac{r}{2 \cdot \pi} \rfloor.$$

$$(26) \quad \text{If } \cos r = 1, \text{ then } r = 2 \cdot \pi \cdot \lfloor \frac{r}{2 \cdot \pi} \rfloor.$$

$$(27) \quad \text{If } 0 \leq r \text{ and } r \leq 2 \cdot \pi \text{ and } \sin r = -1, \text{ then } r = \frac{3}{2} \cdot \pi.$$

$$(28) \quad \text{If } 0 \leq r \text{ and } r \leq 2 \cdot \pi \text{ and } \sin r = 1, \text{ then } r = \frac{\pi}{2}.$$

$$(29) \quad \text{If } 0 \leq r \text{ and } r \leq 2 \cdot \pi \text{ and } \cos r = -1, \text{ then } r = \pi.$$

$$(30) \quad \text{If } 0 \leq r \text{ and } r < \frac{\pi}{2}, \text{ then } \sin r < 1.$$

$$(31) \quad \text{If } 0 \leq r \text{ and } r < \frac{3}{2} \cdot \pi, \text{ then } \sin r > -1.$$

$$(32) \quad \text{If } \frac{3}{2} \cdot \pi < r \text{ and } r \leq 2 \cdot \pi, \text{ then } \sin r > -1.$$

$$(33) \quad \text{If } \frac{\pi}{2} < r \text{ and } r \leq 2 \cdot \pi, \text{ then } \sin r < 1.$$

$$(34) \quad \text{If } 0 < r \text{ and } r < 2 \cdot \pi, \text{ then } \cos r < 1.$$

$$(35) \quad \text{If } 0 \leq r \text{ and } r < \pi, \text{ then } \cos r > -1.$$

$$(36) \quad \text{If } \pi < r \text{ and } r \leq 2 \cdot \pi, \text{ then } \cos r > -1.$$

$$(37) \quad \text{If } 2 \cdot \pi \cdot i \leq r \text{ and } r < \frac{\pi}{2} + 2 \cdot \pi \cdot i, \text{ then } \sin r < 1.$$

- (38) If $2 \cdot \pi \cdot i \leq r$ and $r < \frac{3}{2} \cdot \pi + 2 \cdot \pi \cdot i$, then $\sin r > -1$.
- (39) If $\frac{3}{2} \cdot \pi + 2 \cdot \pi \cdot i < r$ and $r \leq 2 \cdot \pi + 2 \cdot \pi \cdot i$, then $\sin r > -1$.
- (40) If $\frac{\pi}{2} + 2 \cdot \pi \cdot i < r$ and $r \leq 2 \cdot \pi + 2 \cdot \pi \cdot i$, then $\sin r < 1$.
- (41) If $2 \cdot \pi \cdot i < r$ and $r < 2 \cdot \pi + 2 \cdot \pi \cdot i$, then $\cos r < 1$.
- (42) If $2 \cdot \pi \cdot i \leq r$ and $r < \pi + 2 \cdot \pi \cdot i$, then $\cos r > -1$.
- (43) If $\pi + 2 \cdot \pi \cdot i < r$ and $r \leq 2 \cdot \pi + 2 \cdot \pi \cdot i$, then $\cos r > -1$.
- (44) If $\cos(2 \cdot \pi \cdot r) = 1$, then $r \in \mathbb{Z}$.
- (45) (The function $\sin$) $^{\circ}[-\frac{\pi}{2}, \frac{\pi}{2}] = [-1, 1]$.
- (46) (The function $\sin$) $^{\circ}] -\frac{\pi}{2}, \frac{\pi}{2}[=]-1, 1[$.
- (47) (The function $\sin$) $^{\circ}[\frac{\pi}{2}, \frac{3}{2} \cdot \pi] = [-1, 1]$.
- (48) (The function $\sin$) $^{\circ}]\frac{\pi}{2}, \frac{3}{2} \cdot \pi[=]-1, 1[$.
- (49) (The function $\cos$) $^{\circ}[0, \pi] = [-1, 1]$.
- (50) (The function $\cos$) $^{\circ}]0, \pi[=]-1, 1[$.
- (51) (The function $\cos$) $^{\circ}[\pi, 2 \cdot \pi] = [-1, 1]$.
- (52) (The function $\cos$) $^{\circ}]\pi, 2 \cdot \pi[=]-1, 1[$.
- (53) The function $\sin$ is increasing on $[-\frac{\pi}{2} + 2 \cdot \pi \cdot i, \frac{\pi}{2} + 2 \cdot \pi \cdot i]$.
- (54) The function $\sin$ is decreasing on $[\frac{\pi}{2} + 2 \cdot \pi \cdot i, \frac{3}{2} \cdot \pi + 2 \cdot \pi \cdot i]$.
- (55) The function $\cos$ is decreasing on $[2 \cdot \pi \cdot i, \pi + 2 \cdot \pi \cdot i]$.
- (56) The function $\cos$ is increasing on $[\pi + 2 \cdot \pi \cdot i, 2 \cdot \pi + 2 \cdot \pi \cdot i]$.
- (57) (The function $\sin$) $\upharpoonright[-\frac{\pi}{2} + 2 \cdot \pi \cdot i, \frac{\pi}{2} + 2 \cdot \pi \cdot i]$ is one-to-one.
- (58) (The function $\sin$) $\upharpoonright[\frac{\pi}{2} + 2 \cdot \pi \cdot i, \frac{3}{2} \cdot \pi + 2 \cdot \pi \cdot i]$ is one-to-one.

One can check that (the function $\sin$) $\upharpoonright[-\frac{\pi}{2}, \frac{\pi}{2}]$ is one-to-one and (the function $\sin$) $\upharpoonright[\frac{\pi}{2}, \frac{3}{2} \cdot \pi]$ is one-to-one.

One can check the following observations:

- * (the function $\sin$) $\upharpoonright[-\frac{\pi}{2}, 0]$ is one-to-one,
- * (the function $\sin$) $\upharpoonright[0, \frac{\pi}{2}]$ is one-to-one,
- * (the function $\sin$) $\upharpoonright[\frac{\pi}{2}, \pi]$ is one-to-one,
- * (the function $\sin$) $\upharpoonright[\pi, \frac{3}{2} \cdot \pi]$ is one-to-one, and
- * (the function $\sin$) $\upharpoonright[\frac{3}{2} \cdot \pi, 2 \cdot \pi]$ is one-to-one.

One can verify the following observations:

- * (the function $\sin$) $\upharpoonright]-\frac{\pi}{2}, \frac{\pi}{2}[$ is one-to-one,
- * (the function $\sin$) $\upharpoonright]\frac{\pi}{2}, \frac{3}{2} \cdot \pi[$ is one-to-one,
- * (the function $\sin$) $\upharpoonright]-\frac{\pi}{2}, 0[$ is one-to-one,
- * (the function $\sin$) $\upharpoonright]0, \frac{\pi}{2}[$ is one-to-one,
- * (the function $\sin$) $\upharpoonright]\frac{\pi}{2}, \pi[$ is one-to-one,
- * (the function $\sin$) $\upharpoonright]\pi, \frac{3}{2} \cdot \pi[$ is one-to-one, and
- * (the function $\sin$) $\upharpoonright]\frac{3}{2} \cdot \pi, 2 \cdot \pi[$ is one-to-one.

Next we state two propositions:

(59) (The function $\cos$) $\restriction[2 \cdot \pi \cdot i, \pi + 2 \cdot \pi \cdot i]$ is one-to-one.

(60) (The function $\cos$) $\restriction[\pi + 2 \cdot \pi \cdot i, 2 \cdot \pi + 2 \cdot \pi \cdot i]$ is one-to-one.

Let us note that (the function $\cos$) $\restriction[0, \pi]$ is one-to-one and (the function $\cos$) $\restriction[\pi, 2 \cdot \pi]$ is one-to-one.

One can check the following observations:

- * (the function $\cos$) $\restriction[0, \frac{\pi}{2}]$ is one-to-one,
- * (the function $\cos$) $\restriction[\frac{\pi}{2}, \pi]$ is one-to-one,
- * (the function $\cos$) $\restriction[\pi, \frac{3}{2} \cdot \pi]$ is one-to-one, and
- * (the function $\cos$) $\restriction[\frac{3}{2} \cdot \pi, 2 \cdot \pi]$ is one-to-one.

One can check the following observations:

- * (the function $\cos$) $\restriction]0, \pi[$ is one-to-one,
- * (the function $\cos$) $\restriction]\pi, 2 \cdot \pi[$ is one-to-one,
- * (the function $\cos$) $\restriction]0, \frac{\pi}{2}[$ is one-to-one,
- * (the function $\cos$) $\restriction]\frac{\pi}{2}, \pi[$ is one-to-one,
- * (the function $\cos$) $\restriction]\pi, \frac{3}{2} \cdot \pi[$ is one-to-one, and
- * (the function $\cos$) $\restriction]\frac{3}{2} \cdot \pi, 2 \cdot \pi[$ is one-to-one.

The following proposition is true

(61) If $2 \cdot \pi \cdot i \leq r$ and $r < 2 \cdot \pi + 2 \cdot \pi \cdot i$ and $2 \cdot \pi \cdot i \leq s$ and $s < 2 \cdot \pi + 2 \cdot \pi \cdot i$ and $\sin r = \sin s$ and $\cos r = \cos s$, then $r = s$.

3. FUNCTION ARCSIN

The function $\arcsin$ is a partial function from $\mathbb{R}$ to $\mathbb{R}$ and is defined by:

(Def. 1) The function $\arcsin = ((\text{the function } \sin)\restriction[-\frac{\pi}{2}, \frac{\pi}{2}])^{-1}$.

Let r be a set. The functor $\arcsin r$ is defined by:

(Def. 2) $\arcsin r = (\text{the function } \arcsin)(r)$.

Let r be a set. Then $\arcsin r$ is a real number.

Next we state two propositions:

(62) $(\text{The function } \arcsin)^{-1} = (\text{the function } \sin)\restriction[-\frac{\pi}{2}, \frac{\pi}{2}]$.

(63) $\text{rng}(\text{the function } \arcsin) = [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Let us note that the function $\arcsin$ is one-to-one.

The following propositions are true:

(64) $\text{dom}(\text{the function } \arcsin) = [-1, 1]$.

(65) $((\text{The function } \sin)\restriction[-\frac{\pi}{2}, \frac{\pi}{2}] \text{ **qua** function}) \cdot (\text{the function } \arcsin) = \text{id}_{[-1, 1]}$.

(66) $(\text{The function } \arcsin) \cdot ((\text{the function } \sin)\restriction[-\frac{\pi}{2}, \frac{\pi}{2}]) = \text{id}_{[-1, 1]}$.

- (67) $((\text{The function } \sin) \upharpoonright [-\frac{\pi}{2}, \frac{\pi}{2}]) \cdot (\text{the function } \arcsin) = \text{id}_{[-\frac{\pi}{2}, \frac{\pi}{2}]}$.
- (68) $(\text{The function } \arcsin \text{ qua function}) \cdot ((\text{the function } \sin) \upharpoonright [-\frac{\pi}{2}, \frac{\pi}{2}]) = \text{id}_{[-\frac{\pi}{2}, \frac{\pi}{2}]}$.
- (69) If $-1 \leq r$ and $r \leq 1$, then $\sin \arcsin r = r$.
- (70) If $-\frac{\pi}{2} \leq r$ and $r \leq \frac{\pi}{2}$, then $\arcsin \sin r = r$.
- (71) $\arcsin(-1) = -\frac{\pi}{2}$.
- (72) $\arcsin 0 = 0$.
- (73) $\arcsin 1 = \frac{\pi}{2}$.
- (74) If $-1 \leq r$ and $r \leq 1$ and $\arcsin r = -\frac{\pi}{2}$, then $r = -1$.
- (75) If $-1 \leq r$ and $r \leq 1$ and $\arcsin r = 0$, then $r = 0$.
- (76) If $-1 \leq r$ and $r \leq 1$ and $\arcsin r = \frac{\pi}{2}$, then $r = 1$.
- (77) If $-1 \leq r$ and $r \leq 1$, then $-\frac{\pi}{2} \leq \arcsin r$ and $\arcsin r \leq \frac{\pi}{2}$.
- (78) If $-1 < r$ and $r < 1$, then $-\frac{\pi}{2} < \arcsin r$ and $\arcsin r < \frac{\pi}{2}$.
- (79) If $-1 \leq r$ and $r \leq 1$, then $\arcsin r = -\arcsin(-r)$.
- (80) If $0 \leq s$ and $r^2 + s^2 = 1$, then $\cos \arcsin r = s$.
- (81) If $s \leq 0$ and $r^2 + s^2 = 1$, then $\cos \arcsin r = -s$.
- (82) If $-1 \leq r$ and $r \leq 1$, then $\cos \arcsin r = \sqrt{1 - r^2}$.
- (83) The function $\arcsin$ is increasing on $[-1, 1]$.
- (84) The function $\arcsin$ is differentiable on $] -1, 1[$ and if $-1 < r$ and $r < 1$, then $(\text{the function } \arcsin)'(r) = \frac{1}{\sqrt{1 - r^2}}$.
- (85) The function $\arcsin$ is continuous on $[-1, 1]$.

4. FUNCTION ARCCOS

The function $\arccos$ is a partial function from $\mathbb{R}$ to $\mathbb{R}$ and is defined by:

(Def. 3) The function $\arccos = ((\text{the function } \cos) \upharpoonright [0, \pi])^{-1}$.

Let r be a set. The functor $\arccos r$ is defined by:

(Def. 4) $\arccos r = (\text{the function } \arccos)(r)$.

Let r be a set. Then $\arccos r$ is a real number.

One can prove the following two propositions:

- (86) $(\text{The function } \arccos)^{-1} = (\text{the function } \cos) \upharpoonright [0, \pi]$.
- (87) $\text{rng}(\text{the function } \arccos) = [0, \pi]$.

Let us note that the function $\arccos$ is one-to-one.

The following propositions are true:

- (88) $\text{dom}(\text{the function } \arccos) = [-1, 1]$.
- (89) $((\text{The function } \cos) \upharpoonright [0, \pi] \text{ qua function}) \cdot (\text{the function } \arccos) = \text{id}_{[-1, 1]}$.
- (90) $(\text{The function } \arccos) \cdot ((\text{the function } \cos) \upharpoonright [0, \pi]) = \text{id}_{[-1, 1]}$.

- (91) $((\text{The function } \cos) \upharpoonright [0, \pi]) \cdot (\text{the function } \arccos) = \text{id}_{[0, \pi]}.$
- (92) $(\text{The function } \arccos \text{ qua function}) \cdot ((\text{the function } \cos) \upharpoonright [0, \pi]) = \text{id}_{[0, \pi]}.$
- (93) If $-1 \leq r$ and $r \leq 1$, then $\cos \arccos r = r.$
- (94) If $0 \leq r$ and $r \leq \pi$, then $\arccos \cos r = r.$
- (95) $\arccos(-1) = \pi.$
- (96) $\arccos 0 = \frac{\pi}{2}.$
- (97) $\arccos 1 = 0.$
- (98) If $-1 \leq r$ and $r \leq 1$ and $\arccos r = 0$, then $r = 1.$
- (99) If $-1 \leq r$ and $r \leq 1$ and $\arccos r = \frac{\pi}{2}$, then $r = 0.$
- (100) If $-1 \leq r$ and $r \leq 1$ and $\arccos r = \pi$, then $r = -1.$
- (101) If $-1 \leq r$ and $r \leq 1$, then $0 \leq \arccos r$ and $\arccos r \leq \pi.$
- (102) If $-1 < r$ and $r < 1$, then $0 < \arccos r$ and $\arccos r < \pi.$
- (103) If $-1 \leq r$ and $r \leq 1$, then $\arccos r = \pi - \arccos(-r).$
- (104) If $0 \leq s$ and $r^2 + s^2 = 1$, then $\sin \arccos r = s.$
- (105) If $s \leq 0$ and $r^2 + s^2 = 1$, then $\sin \arccos r = -s.$
- (106) If $-1 \leq r$ and $r \leq 1$, then $\sin \arccos r = \sqrt{1 - r^2}.$
- (107) The function $\arccos$ is decreasing on $[-1, 1].$
- (108) The function $\arccos$ is differentiable on $] -1, 1[$ and if $-1 < r$ and $r < 1$,
then $(\text{the function } \arccos)'(r) = -\frac{1}{\sqrt{1-r^2}}.$
- (109) The function $\arccos$ is continuous on $[-1, 1].$
- (110) If $-1 \leq r$ and $r \leq 1$, then $\arcsin r + \arccos r = \frac{\pi}{2}.$
- (111) If $-1 \leq r$ and $r \leq 1$, then $\arccos(-r) - \arcsin r = \frac{\pi}{2}.$
- (112) If $-1 \leq r$ and $r \leq 1$, then $\arccos r - \arcsin(-r) = \frac{\pi}{2}.$

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