

Lebesgue Integral of Simple Valued Function¹

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Summary. In this article, the authors introduce Lebesgue integral of simple valued function.

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The terminology and notation used in this paper are introduced in the following papers: [23], [12], [25], [21], [26], [10], [11], [3], [22], [24], [7], [14], [1], [2], [20], [4], [5], [6], [8], [9], [19], [13], [15], [16], [17], and [18].

1. INTEGRAL OF SIMPLE VALUED FUNCTION

The following propositions are true:

- (1) Let n, m be natural numbers, a be a function from $[\text{Seg } n, \text{Seg } m]$ into $\mathbb{R}$, and p, q be finite sequences of elements of $\mathbb{R}$. Suppose that
 - (i) $\text{dom } p = \text{Seg } n$,
 - (ii) for every natural number i such that $i \in \text{dom } p$ there exists a finite sequence r of elements of $\mathbb{R}$ such that $\text{dom } r = \text{Seg } m$ and $p(i) = \sum r$ and for every natural number j such that $j \in \text{dom } r$ holds $r(j) = a(\langle i, j \rangle)$,
 - (iii) $\text{dom } q = \text{Seg } m$, and
 - (iv) for every natural number j such that $j \in \text{dom } q$ there exists a finite sequence s of elements of $\mathbb{R}$ such that $\text{dom } s = \text{Seg } n$ and $q(j) = \sum s$ and for every natural number i such that $i \in \text{dom } s$ holds $s(i) = a(\langle i, j \rangle)$.

Then $\sum p = \sum q$.

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- (2) Let F be a finite sequence of elements of $\overline{\mathbb{R}}$ and f be a finite sequence of elements of $\mathbb{R}$. If $F = f$, then $\sum F = \sum f$.
- (3) Let X be a non empty set, S be a σ -field of subsets of X , and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S . Then there exists a finite sequence F of separated subsets of S and there exists a finite sequence a of elements of $\overline{\mathbb{R}}$ such that
 - (i) $\text{dom } f = \bigcup \text{rng } F$,
 - (ii) $\text{dom } F = \text{dom } a$,
 - (iii) for every natural number n such that $n \in \text{dom } F$ and for every set x such that $x \in F(n)$ holds $f(x) = a(n)$, and
 - (iv) for every set x such that $x \in \text{dom } f$ there exists a finite sequence a_1 of elements of $\overline{\mathbb{R}}$ such that $\text{dom } a_1 = \text{dom } a$ and for every natural number n such that $n \in \text{dom } a_1$ holds $a_1(n) = a(n) \cdot \chi_{F(n), X}(x)$.
- (4) Let X be a set and F be a finite sequence of elements of X . Then F is disjoint valued if and only if for all natural numbers i, j such that $i \in \text{dom } F$ and $j \in \text{dom } F$ and $i \neq j$ holds $F(i)$ misses $F(j)$.
- (5) Let X be a non empty set, A be a set, S be a σ -field of subsets of X , F be a finite sequence of separated subsets of S , and G be a finite sequence of elements of S . Suppose $\text{dom } G = \text{dom } F$ and for every natural number i such that $i \in \text{dom } G$ holds $G(i) = A \cap F(i)$. Then G is a finite sequence of separated subsets of S .
- (6) Let X be a non empty set, A be a set, and F, G be finite sequences of elements of X . Suppose $\text{dom } G = \text{dom } F$ and for every natural number i such that $i \in \text{dom } G$ holds $G(i) = A \cap F(i)$. Then $\bigcup \text{rng } G = A \cap \bigcup \text{rng } F$.
- (7) Let X be a set, F be a finite sequence of elements of X , and i be a natural number. If $i \in \text{dom } F$, then $F(i) \subseteq \bigcup \text{rng } F$ and $F(i) \cap \bigcup \text{rng } F = F(i)$.
- (8) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , and F be a finite sequence of separated subsets of S . Then $\text{dom } F = \text{dom}(M \cdot F)$.
- (9) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , and F be a finite sequence of separated subsets of S . Then $M(\bigcup \text{rng } F) = \sum(M \cdot F)$.
- (10) Let F, G be finite sequences of elements of $\overline{\mathbb{R}}$ and a be an extended real number. Suppose that
 - (i) $a \neq +\infty$ and $a \neq -\infty$ or for every natural number i such that $i \in \text{dom } F$ holds $F(i) < 0_{\overline{\mathbb{R}}}$ or for every natural number i such that $i \in \text{dom } F$ holds $0_{\overline{\mathbb{R}}} < F(i)$,
 - (ii) $\text{dom } F = \text{dom } G$, and
 - (iii) for every natural number i such that $i \in \text{dom } G$ holds $G(i) = a \cdot F(i)$.
 Then $\sum G = a \cdot \sum F$.

- (11) Every finite sequence of elements of $\mathbb{R}$ is a finite sequence of elements of $\overline{\mathbb{R}}$.

Let X be a non empty set, let S be a σ -field of subsets of X , let f be a partial function from X to $\overline{\mathbb{R}}$, let F be a finite sequence of separated subsets of S , and let a be a finite sequence of elements of $\overline{\mathbb{R}}$. We say that F and a are re-representation of f if and only if the conditions (Def. 1) are satisfied.

- (Def. 1)(i) $\text{dom } f = \bigcup \text{rng } F$,
(ii) $\text{dom } F = \text{dom } a$, and
(iii) for every natural number n such that $n \in \text{dom } F$ and for every set x such that $x \in F(n)$ holds $f(x) = a(n)$.

One can prove the following propositions:

- (12) Let X be a non empty set, S be a σ -field of subsets of X , and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S . Then there exists a finite sequence F of separated subsets of S and there exists a finite sequence a of elements of $\overline{\mathbb{R}}$ such that F and a are re-representation of f .
- (13) Let X be a non empty set, S be a σ -field of subsets of X , and F be a finite sequence of separated subsets of S . Then there exists a finite sequence G of separated subsets of S such that
- (i) $\bigcup \text{rng } F = \bigcup \text{rng } G$, and
(ii) for every natural number n such that $n \in \text{dom } G$ holds $G(n) \neq \emptyset$ and there exists a natural number m such that $m \in \text{dom } F$ and $F(m) = G(n)$.
- (14) Let X be a non empty set, S be a σ -field of subsets of X , and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S and for every set x such that $x \in \text{dom } f$ holds $0_{\overline{\mathbb{R}}} \leq f(x)$. Then there exists a finite sequence F of separated subsets of S and there exists a finite sequence a of elements of $\overline{\mathbb{R}}$ such that
- (i) F and a are re-representation of f ,
(ii) $a(1) = 0_{\overline{\mathbb{R}}}$, and
(iii) for every natural number n such that $2 \leq n$ and $n \in \text{dom } a$ holds $0_{\overline{\mathbb{R}}} < a(n)$ and $a(n) < +\infty$.
- (15) Let X be a non empty set, S be a σ -field of subsets of X , f be a partial function from X to $\overline{\mathbb{R}}$, F be a finite sequence of separated subsets of S , a be a finite sequence of elements of $\overline{\mathbb{R}}$, and x be an element of X . Suppose F and a are re-representation of f and $x \in \text{dom } f$. Then there exists a finite sequence a_1 of elements of $\overline{\mathbb{R}}$ such that $\text{dom } a_1 = \text{dom } a$ and for every natural number n such that $n \in \text{dom } a_1$ holds $a_1(n) = a(n) \cdot \chi_{F(n), X}(x)$ and $f(x) = \sum a_1$.
- (16) Let p be a finite sequence of elements of $\overline{\mathbb{R}}$ and q be a finite sequence of elements of $\mathbb{R}$. If $p = q$, then $\sum p = \sum q$.

- (17) Let p be a finite sequence of elements of $\overline{\mathbb{R}}$. Suppose for every natural number n such that $n \in \text{dom } p$ holds $0_{\overline{\mathbb{R}}} \leq p(n)$ and there exists a natural number n such that $n \in \text{dom } p$ and $p(n) = +\infty$. Then $\sum p = +\infty$.

Let X be a non empty set, let S be a σ -field of subsets of X , let M be a σ -measure on S , and let f be a partial function from X to $\overline{\mathbb{R}}$. Let us assume that f is simple function in S and $\text{dom } f \neq \emptyset$ and for every set x such that $x \in \text{dom } f$ holds $0_{\overline{\mathbb{R}}} \leq f(x)$. The functor $\text{integral}(X, S, M, f)$ yielding an element of $\overline{\mathbb{R}}$ is defined by the condition (Def. 2).

- (Def. 2) There exists a finite sequence F of separated subsets of S and there exist finite sequences a, x of elements of $\overline{\mathbb{R}}$ such that
- (i) F and a are re-representation of f ,
 - (ii) $a(1) = 0_{\overline{\mathbb{R}}}$,
 - (iii) for every natural number n such that $2 \leq n$ and $n \in \text{dom } a$ holds $0_{\overline{\mathbb{R}}} < a(n)$ and $a(n) < +\infty$,
 - (iv) $\text{dom } x = \text{dom } F$,
 - (v) for every natural number n such that $n \in \text{dom } x$ holds $x(n) = a(n) \cdot (M \cdot F)(n)$, and
 - (vi) $\text{integral}(X, S, M, f) = \sum x$.

2. ADDITIONAL LEMMA

We now state the proposition

- (18) Let a be a finite sequence of elements of $\overline{\mathbb{R}}$ and p, N be elements of $\overline{\mathbb{R}}$. Suppose $N = \text{len } a$ and for every natural number n such that $n \in \text{dom } a$ holds $a(n) = p$. Then $\sum a = N \cdot p$.

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