

Hölder's Inequality and Minkowski's Inequality

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Summary. In this article, Hölder's inequality and Minkowski's inequality are proved. These equalities are basic ones of functional analysis.

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The papers [12], [13], [14], [3], [1], [11], [4], [2], [7], [5], [6], [10], [8], and [9] provide the notation and terminology for this paper.

1. HÖLDER'S INEQUALITY

In this paper a, b, p, q are real numbers.

Let x be a real number. One can verify that $[x, +\infty[$ is non empty.

Next we state several propositions:

- (1) For all real numbers p, q such that $0 < p$ and $0 < q$ and for every real number a such that $0 \leq a$ holds $a^p \cdot a^q = a^{p+q}$.
- (2) For all real numbers p, q such that $0 < p$ and $0 < q$ and for every real number a such that $0 \leq a$ holds $(a^p)^q = a^{p \cdot q}$.
- (3) For every real number p such that $0 < p$ and for all real numbers a, b such that $0 \leq a$ and $a \leq b$ holds $a^p \leq b^p$.
- (4) If $1 < p$ and $\frac{1}{p} + \frac{1}{q} = 1$ and $0 < a$ and $0 < b$, then $a \cdot b \leq \frac{a^p}{p} + \frac{b^q}{q}$ and $a \cdot b = \frac{a^p}{p} + \frac{b^q}{q}$ iff $a^{\frac{p}{p-1}} = b^{\frac{q}{q-1}}$.
- (5) If $1 < p$ and $\frac{1}{p} + \frac{1}{q} = 1$ and $0 \leq a$ and $0 \leq b$, then $a \cdot b \leq \frac{a^p}{p} + \frac{b^q}{q}$ and $a \cdot b = \frac{a^p}{p} + \frac{b^q}{q}$ iff $a^p = b^q$.

2. MINKOWSKI'S INEQUALITY

Next we state several propositions:

- (6) Let p, q be real numbers. Suppose $1 < p$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let a, b, a_1, b_1, a_2 be sequences of real numbers. Suppose that for every natural number n holds $a_1(n) = |a(n)|^p$ and $b_1(n) = |b(n)|^q$ and $a_2(n) = |a(n) \cdot b(n)|$. Let n be a natural number. Then $(\sum_{\alpha=0}^{\kappa} (a_2)(\alpha))_{\kappa \in \mathbb{N}}(n) \leq (\sum_{\alpha=0}^{\kappa} (a_1)(\alpha))_{\kappa \in \mathbb{N}}(n)^{\frac{1}{p}} \cdot (\sum_{\alpha=0}^{\kappa} (b_1)(\alpha))_{\kappa \in \mathbb{N}}(n)^{\frac{1}{q}}$.
- (7) Let p be a real number. Suppose $1 < p$. Let a, b, a_1, b_2, a_2 be sequences of real numbers. Suppose that for every natural number n holds $a_1(n) = |a(n)|^p$ and $b_2(n) = |b(n)|^p$ and $a_2(n) = |a(n) + b(n)|^p$. Let n be a natural number. Then $(\sum_{\alpha=0}^{\kappa} (a_2)(\alpha))_{\kappa \in \mathbb{N}}(n)^{\frac{1}{p}} \leq (\sum_{\alpha=0}^{\kappa} (a_1)(\alpha))_{\kappa \in \mathbb{N}}(n)^{\frac{1}{p}} + (\sum_{\alpha=0}^{\kappa} (b_2)(\alpha))_{\kappa \in \mathbb{N}}(n)^{\frac{1}{p}}$.
- (8) Let a, b be sequences of real numbers. Suppose for every natural number n holds $a(n) \leq b(n)$ and b is convergent and a is non-decreasing. Then a is convergent and $\lim a \leq \lim b$.
- (9) Let a, b, c be sequences of real numbers. Suppose for every natural number n holds $a(n) \leq b(n) + c(n)$ and b is convergent and c is convergent and a is non-decreasing. Then a is convergent and $\lim a \leq \lim b + \lim c$.
- (10) Let p be a real number. Suppose $0 < p$. Let a, a_1 be sequences of real numbers. Suppose a is convergent and for every natural number n holds $0 \leq a(n)$ and for every natural number n holds $a_1(n) = a(n)^p$. Then a_1 is convergent and $\lim a_1 = (\lim a)^p$.
- (11) Let p be a real number. Suppose $0 < p$. Let a, a_1 be sequences of real numbers. Suppose a is summable and for every natural number n holds $0 \leq a(n)$ and for every natural number n holds $a_1(n) = (\sum_{\alpha=0}^{\kappa} a(\alpha))_{\kappa \in \mathbb{N}}(n)^p$. Then a_1 is convergent and $\lim a_1 = (\sum a)^p$ and a_1 is non-decreasing and for every natural number n holds $a_1(n) \leq (\sum a)^p$.
- (12) Let p, q be real numbers. Suppose $1 < p$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let a, b, a_1, b_1, a_2 be sequences of real numbers. Suppose for every natural number n holds $a_1(n) = |a(n)|^p$ and $b_1(n) = |b(n)|^q$ and $a_2(n) = |a(n) \cdot b(n)|$ and a_1 is summable and b_1 is summable. Then a_2 is summable and $\sum a_2 \leq (\sum a_1)^{\frac{1}{p}} \cdot (\sum b_1)^{\frac{1}{q}}$.
- (13) Let p be a real number. Suppose $1 < p$. Let a, b, a_1, b_2, a_2 be sequences of real numbers. Suppose that
 - (i) for every natural number n holds $a_1(n) = |a(n)|^p$ and $b_2(n) = |b(n)|^p$ and $a_2(n) = |a(n) + b(n)|^p$,
 - (ii) a_1 is summable, and
 - (iii) b_2 is summable.

Then a_2 is summable and $(\sum a_2)^{\frac{1}{p}} \leq (\sum a_1)^{\frac{1}{p}} + (\sum b_2)^{\frac{1}{p}}$.

REFERENCES

- [1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(1):91–96, 1990.
- [2] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(3):507–513, 1990.
- [3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [4] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(1):35–40, 1990.
- [5] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(2):273–275, 1990.
- [6] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1(3):471–475, 1990.
- [7] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(2):269–272, 1990.
- [8] Konrad Raczkowski. Integer and rational exponents. *Formalized Mathematics*, 2(1):125–130, 1991.
- [9] Konrad Raczkowski and Andrzej Nędzusiak. Real exponents and logarithms. *Formalized Mathematics*, 2(2):213–216, 1991.
- [10] Konrad Raczkowski and Andrzej Nędzusiak. Series. *Formalized Mathematics*, 2(4):449–452, 1991.
- [11] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics*.
- [12] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(1):9–11, 1990.
- [13] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [14] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.

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