

Several Differentiation Formulas of Special Functions. Part IV

Bo Li
 Qingdao University of Science
 and Technology
 China

Peng Wang
 Qingdao University of Science
 and Technology
 China

Summary. In this article, we give several differentiation formulas of special and composite functions including trigonometric function, polynomial function and logarithmic function.

MML identifier: FDIFF_8, version: 7.8.03 4.75.958

The notation and terminology used here are introduced in the following papers: [13], [15], [1], [16], [2], [4], [10], [11], [17], [5], [14], [12], [3], [7], [6], [9], and [8].

For simplicity, we adopt the following convention: x , a , b , c denote real numbers, n denotes a natural number, Z denotes an open subset of $\mathbb{R}$, and f , f_1 , f_2 denote partial functions from $\mathbb{R}$ to $\mathbb{R}$.

Next we state a number of propositions:

- (1) If $x \in \text{dom}(\text{the function } \tan)$, then $(\text{the function } \cos)(x) \neq 0$.
- (2) If $x \in \text{dom}(\text{the function } \cot)$, then $(\text{the function } \sin)(x) \neq 0$.
- (3) If $Z \subseteq \text{dom}(\frac{f_1}{f_2})$, then for every x such that $x \in Z$ holds $(\frac{f_1}{f_2})(x)_{\mathbb{Z}}^n = \frac{f_1(x)_{\mathbb{Z}}^n}{f_2(x)_{\mathbb{Z}}^n}$.
- (4) Suppose $Z \subseteq \text{dom}(\frac{f_1}{f_2})$ and for every x such that $x \in Z$ holds $f_1(x) = x + a$ and $f_2(x) = x - b$. Then $\frac{f_1}{f_2}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{f_1}{f_2})'_{|Z}(x) = \frac{-a-b}{(x-b)^2}$.
- (5) Suppose $Z \subseteq \text{dom}((\text{the function } \ln) \cdot \frac{1}{f})$ and for every x such that $x \in Z$ holds $f(x) = x$. Then $(\text{the function } \ln) \cdot \frac{1}{f}$ is differentiable on Z and for every x such that $x \in Z$ holds $((\text{the function } \ln) \cdot \frac{1}{f})'_{|Z}(x) = -\frac{1}{x}$.
- (6) Suppose $Z \subseteq \text{dom}((\text{the function } \tan) \cdot f)$ and for every x such that $x \in Z$ holds $f(x) = a \cdot x + b$. Then

- (i) (the function $\tan$) $\cdot f$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $((\text{the function } \tan) \cdot f)'_{|Z}(x) = \frac{a}{(\text{the function } \cos)(a \cdot x + b)^2}$.
- (7) Suppose $Z \subseteq \text{dom}((\text{the function } \cot) \cdot f)$ and for every x such that $x \in Z$ holds $f(x) = a \cdot x + b$. Then
 - (i) (the function $\cot$) $\cdot f$ is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $((\text{the function } \cot) \cdot f)'_{|Z}(x) = -\frac{a}{(\text{the function } \sin)(a \cdot x + b)^2}$.
- (8) Suppose $Z \subseteq \text{dom}((\text{the function } \tan) \cdot \frac{1}{f})$ and for every x such that $x \in Z$ holds $f(x) = x$. Then
 - (i) (the function $\tan$) $\cdot \frac{1}{f}$ is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $((\text{the function } \tan) \cdot \frac{1}{f})'_{|Z}(x) = -\frac{1}{x^2 \cdot (\text{the function } \cos)(\frac{1}{x})^2}$.
- (9) Suppose $Z \subseteq \text{dom}((\text{the function } \cot) \cdot \frac{1}{f})$ and for every x such that $x \in Z$ holds $f(x) = x$. Then
 - (i) (the function $\cot$) $\cdot \frac{1}{f}$ is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $((\text{the function } \cot) \cdot \frac{1}{f})'_{|Z}(x) = \frac{1}{x^2 \cdot (\text{the function } \sin)(\frac{1}{x})^2}$.
- (10) Suppose $Z \subseteq \text{dom}((\text{the function } \tan) \cdot (f_1 + c f_2))$ and $f_2 = \frac{2}{\mathbb{Z}}$ and for every x such that $x \in Z$ holds $f_1(x) = a + b \cdot x$. Then
 - (i) (the function $\tan$) $\cdot (f_1 + c f_2)$ is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $((\text{the function } \tan) \cdot (f_1 + c f_2))'_{|Z}(x) = \frac{b+2 \cdot c \cdot x}{(\text{the function } \cos)(a+b \cdot x+c \cdot x^2)^2}$.
- (11) Suppose $Z \subseteq \text{dom}((\text{the function } \cot) \cdot (f_1 + c f_2))$ and $f_2 = \frac{2}{\mathbb{Z}}$ and for every x such that $x \in Z$ holds $f_1(x) = a + b \cdot x$. Then
 - (i) (the function $\cot$) $\cdot (f_1 + c f_2)$ is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $((\text{the function } \cot) \cdot (f_1 + c f_2))'_{|Z}(x) = -\frac{b+2 \cdot c \cdot x}{(\text{the function } \sin)(a+b \cdot x+c \cdot x^2)^2}$.
- (12) Suppose $Z \subseteq \text{dom}((\text{the function } \tan) \cdot (\text{the function } \exp))$. Then
 - (i) (the function $\tan$) $\cdot (\text{the function } \exp)$ is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $((\text{the function } \tan) \cdot (\text{the function } \exp))'_{|Z}(x) = \frac{(\text{the function } \exp)(x)}{(\text{the function } \cos)((\text{the function } \exp)(x))^2}$.
- (13) Suppose $Z \subseteq \text{dom}((\text{the function } \cot) \cdot (\text{the function } \exp))$. Then
 - (i) (the function $\cot$) $\cdot (\text{the function } \exp)$ is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $((\text{the function } \cot) \cdot (\text{the function } \exp))'_{|Z}(x) = -\frac{(\text{the function } \exp)(x)}{(\text{the function } \sin)((\text{the function } \exp)(x))^2}$.
- (14) Suppose $Z \subseteq \text{dom}((\text{the function } \tan) \cdot (\text{the function } \ln))$. Then
 - (i) (the function $\tan$) $\cdot (\text{the function } \ln)$ is differentiable on Z , and

- (ii) for every x such that $x \in Z$ holds $((\text{the function tan}) \cdot (\text{the function ln}))'_{|Z}(x) = \frac{1}{x \cdot (\text{the function cos})((\text{the function ln})(x))^2}$.
- (15) Suppose $Z \subseteq \text{dom}((\text{the function cot}) \cdot (\text{the function ln}))$. Then
- (i) $(\text{the function cot}) \cdot (\text{the function ln})$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $((\text{the function cot}) \cdot (\text{the function ln}))'_{|Z}(x) = -\frac{1}{x \cdot (\text{the function sin})((\text{the function ln})(x))^2}$.
- (16) Suppose $Z \subseteq \text{dom}((\text{the function exp}) \cdot (\text{the function tan}))$. Then
- (i) $(\text{the function exp}) \cdot (\text{the function tan})$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $((\text{the function exp}) \cdot (\text{the function tan}))'_{|Z}(x) = \frac{(\text{the function exp})((\text{the function tan})(x))}{(\text{the function cos})(x)^2}$.
- (17) Suppose $Z \subseteq \text{dom}((\text{the function exp}) \cdot (\text{the function cot}))$. Then
- (i) $(\text{the function exp}) \cdot (\text{the function cot})$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $((\text{the function exp}) \cdot (\text{the function cot}))'_{|Z}(x) = -\frac{(\text{the function exp})((\text{the function cot})(x))}{(\text{the function sin})(x)^2}$.
- (18) Suppose $Z \subseteq \text{dom}((\text{the function ln}) \cdot (\text{the function tan}))$. Then
- (i) $(\text{the function ln}) \cdot (\text{the function tan})$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $((\text{the function ln}) \cdot (\text{the function tan}))'_{|Z}(x) = \frac{1}{(\text{the function cos})(x) \cdot (\text{the function sin})(x)}$.
- (19) Suppose $Z \subseteq \text{dom}((\text{the function ln}) \cdot (\text{the function cot}))$. Then
- (i) $(\text{the function ln}) \cdot (\text{the function cot})$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $((\text{the function ln}) \cdot (\text{the function cot}))'_{|Z}(x) = -\frac{1}{(\text{the function sin})(x) \cdot (\text{the function cos})(x)}$.
- (20) Suppose $Z \subseteq \text{dom}((\frac{n}{Z}) \cdot (\text{the function tan}))$ and $1 \leq n$. Then
- (i) $(\frac{n}{Z}) \cdot (\text{the function tan})$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $((\frac{n}{Z}) \cdot (\text{the function tan}))'_{|Z}(x) = \frac{n \cdot (\text{the function sin})(x) \frac{n-1}{Z}}{(\text{the function cos})(x) \frac{n+1}{Z}}$.
- (21) Suppose $Z \subseteq \text{dom}((\frac{n}{Z}) \cdot (\text{the function cot}))$ and $1 \leq n$. Then
- (i) $(\frac{n}{Z}) \cdot (\text{the function cot})$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $((\frac{n}{Z}) \cdot (\text{the function cot}))'_{|Z}(x) = -\frac{n \cdot (\text{the function cos})(x) \frac{n-1}{Z}}{(\text{the function sin})(x) \frac{n+1}{Z}}$.
- (22) Suppose that
- (i) $Z \subseteq \text{dom}((\text{the function tan}) + \frac{1}{\text{the function cos}})$, and
- (ii) for every x such that $x \in Z$ holds $1 + (\text{the function sin})(x) \neq 0$ and $1 - (\text{the function sin})(x) \neq 0$.
- Then
- (iii) $(\text{the function tan}) + \frac{1}{\text{the function cos}}$ is differentiable on Z , and
- (iv) for every x such that $x \in Z$ holds $((\text{the function tan}) + \frac{1}{\text{the function cos}})'_{|Z}(x) = \frac{1}{1 - (\text{the function sin})(x)}$.

(23) Suppose that

- (i) $Z \subseteq \text{dom}((\text{the function tan}) - \frac{1}{\text{the function cos}})$, and
- (ii) for every x such that $x \in Z$ holds $1 - (\text{the function sin})(x) \neq 0$ and $1 + (\text{the function sin})(x) \neq 0$.

Then

- (iii) $(\text{the function tan}) - \frac{1}{\text{the function cos}}$ is differentiable on Z , and
 - (iv) for every x such that $x \in Z$ holds $((\text{the function tan}) - \frac{1}{\text{the function cos}})'|_Z(x) = \frac{1}{1 + (\text{the function sin})(x)}$.
- (24) Suppose $Z \subseteq \text{dom}((\text{the function tan}) - \text{id}_Z)$. Then
- (i) $(\text{the function tan}) - \text{id}_Z$ is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $((\text{the function tan}) - \text{id}_Z)'|_Z(x) = \frac{(\text{the function sin})(x)^2}{(\text{the function cos})(x)^2}$.
- (25) Suppose $Z \subseteq \text{dom}(-\text{the function cot} - \text{id}_Z)$. Then
- (i) $-\text{the function cot} - \text{id}_Z$ is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $(-\text{the function cot} - \text{id}_Z)'|_Z(x) = \frac{(\text{the function cos})(x)^2}{(\text{the function sin})(x)^2}$.
- (26) Suppose $Z \subseteq \text{dom}(\frac{1}{a}((\text{the function tan}) \cdot f) - \text{id}_Z)$ and for every x such that $x \in Z$ holds $f(x) = a \cdot x$ and $a \neq 0$. Then
- (i) $\frac{1}{a}((\text{the function tan}) \cdot f) - \text{id}_Z$ is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $(\frac{1}{a}((\text{the function tan}) \cdot f) - \text{id}_Z)'|_Z(x) = \frac{(\text{the function sin})(a \cdot x)^2}{(\text{the function cos})(a \cdot x)^2}$.
- (27) Suppose $Z \subseteq \text{dom}((-\frac{1}{a})((\text{the function cot}) \cdot f) - \text{id}_Z)$ and for every x such that $x \in Z$ holds $f(x) = a \cdot x$ and $a \neq 0$. Then
- (i) $(-\frac{1}{a})((\text{the function cot}) \cdot f) - \text{id}_Z$ is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $((-\frac{1}{a})((\text{the function cot}) \cdot f) - \text{id}_Z)'|_Z(x) = \frac{(\text{the function cos})(a \cdot x)^2}{(\text{the function sin})(a \cdot x)^2}$.
- (28) Suppose $Z \subseteq \text{dom}(f(\text{the function tan}))$ and for every x such that $x \in Z$ holds $f(x) = a \cdot x + b$. Then
- (i) $f(\text{the function tan})$ is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $(f(\text{the function tan}))'|_Z(x) = \frac{a \cdot (\text{the function sin})(x)}{(\text{the function cos})(x)} + \frac{a \cdot x + b}{(\text{the function cos})(x)^2}$.
- (29) Suppose $Z \subseteq \text{dom}(f(\text{the function cot}))$ and for every x such that $x \in Z$ holds $f(x) = a \cdot x + b$. Then
- (i) $f(\text{the function cot})$ is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $(f(\text{the function cot}))'|_Z(x) = \frac{a \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)} - \frac{a \cdot x + b}{(\text{the function sin})(x)^2}$.
- (30) Suppose $Z \subseteq \text{dom}((\text{the function exp})(\text{the function tan}))$. Then
- (i) $(\text{the function exp})(\text{the function tan})$ is differentiable on Z , and

- (ii) for every x such that $x \in Z$ holds ((the function exp) (the function tan))' $_{|Z}(x) = \frac{(\text{the function exp})(x) \cdot (\text{the function sin})(x)}{(\text{the function cos})(x)} + \frac{(\text{the function exp})(x)}{(\text{the function cos})(x)^2}$.
- (31) Suppose $Z \subseteq \text{dom}((\text{the function exp}) (\text{the function cot}))$. Then
- (i) (the function exp) (the function cot) is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds ((the function exp) (the function cot))' $_{|Z}(x) = \frac{(\text{the function exp})(x) \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)} - \frac{(\text{the function exp})(x)}{(\text{the function sin})(x)^2}$.
- (32) Suppose $Z \subseteq \text{dom}((\text{the function ln}) (\text{the function tan}))$. Then
- (i) (the function ln) (the function tan) is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds ((the function ln) (the function tan))' $_{|Z}(x) = \frac{\frac{(\text{the function sin})(x)}{(\text{the function cos})(x)}}{x} + \frac{(\text{the function ln})(x)}{(\text{the function cos})(x)^2}$.
- (33) Suppose $Z \subseteq \text{dom}((\text{the function ln}) (\text{the function cot}))$. Then
- (i) (the function ln) (the function cot) is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds ((the function ln) (the function cot))' $_{|Z}(x) = \frac{\frac{(\text{the function cos})(x)}{(\text{the function sin})(x)}}{x} - \frac{(\text{the function ln})(x)}{(\text{the function sin})(x)^2}$.
- (34) Suppose $Z \subseteq \text{dom}(\frac{1}{f} (\text{the function tan}))$ and for every x such that $x \in Z$ holds $f(x) = x$. Then
- (i) $\frac{1}{f}$ (the function tan) is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $(\frac{1}{f} (\text{the function tan}))'_{|Z}(x) = -\frac{\frac{(\text{the function sin})(x)}{(\text{the function cos})(x)}}{x^2} + \frac{\frac{1}{x}}{(\text{the function cos})(x)^2}$.
- (35) Suppose $Z \subseteq \text{dom}(\frac{1}{f} (\text{the function cot}))$ and for every x such that $x \in Z$ holds $f(x) = x$. Then
- (i) $\frac{1}{f}$ (the function cot) is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $(\frac{1}{f} (\text{the function cot}))'_{|Z}(x) = -\frac{\frac{(\text{the function cos})(x)}{(\text{the function sin})(x)}}{x^2} - \frac{\frac{1}{x}}{(\text{the function sin})(x)^2}$.

REFERENCES

- [1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(1):91–96, 1990.
- [2] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(2):357–367, 1990.
- [3] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(1):35–40, 1990.
- [4] Jarosław Kotowicz. Partial functions from a domain to a domain. *Formalized Mathematics*, 1(4):697–702, 1990.
- [5] Jarosław Kotowicz. Partial functions from a domain to the set of real numbers. *Formalized Mathematics*, 1(4):703–709, 1990.
- [6] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(2):269–272, 1990.
- [7] Rafał Kwiatek. Factorial and Newton coefficients. *Formalized Mathematics*, 1(5):887–890, 1990.
- [8] Konrad Raczkowski. Integer and rational exponents. *Formalized Mathematics*, 2(1):125–130, 1991.
- [9] Konrad Raczkowski and Paweł Sadowski. Real function differentiability. *Formalized Mathematics*, 1(4):797–801, 1990.

- [10] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(4):777–780, 1990.
- [11] Yasunari Shidama. The Taylor expansions. *Formalized Mathematics*, 12(2):195–200, 2004.
- [12] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics*.
- [13] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(1):9–11, 1990.
- [14] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(3):445–449, 1990.
- [15] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [16] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(1):181–186, 1990.
- [17] Yuguang Yang and Yasunari Shidama. Trigonometric functions and existence of circle ratio. *Formalized Mathematics*, 7(2):255–263, 1998.

Received September 29, 2006
