

Integral of Measurable Function¹

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Summary. In this paper we construct integral of measurable function.

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The terminology and notation used here are introduced in the following articles: [29], [12], [32], [1], [27], [18], [33], [9], [2], [34], [13], [11], [10], [28], [31], [20], [30], [3], [4], [5], [14], [7], [17], [15], [16], [26], [8], [19], [21], [24], [23], [6], [22], and [25].

1. LEMMAS FOR EXTENDED REAL NUMBERS

One can prove the following propositions:

- (1) For all extended real numbers x, y holds $|x - y| = |y - x|$.
- (2) For all extended real numbers x, y holds $y - x \leq |x - y|$.
- (3) Let x, y be extended real numbers and e be a real number. Suppose $|x - y| < e$ and $x \neq +\infty$ or $y \neq +\infty$ but $x \neq -\infty$ or $y \neq -\infty$. Then $x \neq +\infty$ and $x \neq -\infty$ and $y \neq +\infty$ and $y \neq -\infty$.
- (4) For all extended real numbers x, y such that for every real number e such that $0 < e$ holds $x < y + \overline{\mathbb{R}}(e)$ holds $x \leq y$.
- (5) For all extended real numbers x, y, t such that $t \neq -\infty$ and $t \neq +\infty$ and $x < y$ holds $x + t < y + t$.
- (6) For all extended real numbers x, y, t such that $t \neq -\infty$ and $t \neq +\infty$ and $x < y$ holds $x - t < y - t$.

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- (7) For all real numbers a, b holds $\overline{\mathbb{R}}(a) + \overline{\mathbb{R}}(b) = a + b$ and $-\overline{\mathbb{R}}(a) = -a$.
- (8) Let n be a natural number and p be an extended real number. Suppose $0 \leq p$ and $p < n$. Then there exists a natural number k such that $1 \leq k$ and $k \leq 2^n \cdot n$ and $\frac{k-1}{2^n} \leq p$ and $p < \frac{k}{2^n}$.
- (9) Let n, k be natural numbers and p be an extended real number. If $1 \leq k$ and $k \leq 2^n \cdot n$ and $n \leq p$ and $\frac{k-1}{2^n} \leq p$, then $\frac{k}{2^n} \leq p$.
- (10) For all extended real numbers x, y, w, z such that $-\infty < w$ holds if $x < y$ and $w < z$, then $x + w < y + z$.
- (11) For all extended real numbers x, y, k such that $0 \leq k$ holds $k \cdot \max(x, y) = \max(k \cdot x, k \cdot y)$ and $k \cdot \min(x, y) = \min(k \cdot x, k \cdot y)$.
- (12) For all extended real numbers x, y, k such that $k \leq 0$ holds $k \cdot \min(x, y) = \max(k \cdot x, k \cdot y)$ and $k \cdot \max(x, y) = \min(k \cdot x, k \cdot y)$.
- (13) For all extended real numbers x, y, z such that $0 \leq x$ and $0 \leq z$ and $z + x \leq y$ holds $z \leq y$.

2. LEMMAS FOR PARTIAL FUNCTION OF NON-EMPTY SET, EXTENDED REAL NUMBERS

Let I_1 be a set. We say that I_1 is non-positive if and only if:

- (Def. 1) For every extended real number x such that $x \in I_1$ holds $x \leq 0$.

Let R be a binary relation. We say that R is non-positive if and only if:

- (Def. 2) $\text{rng } R$ is non-positive.

The following propositions are true:

- (14) Let X be a set and F be a partial function from X to $\overline{\mathbb{R}}$. Then F is non-positive if and only if for every set n holds $F(n) \leq 0_{\overline{\mathbb{R}}}$.
- (15) Let X be a set and F be a partial function from X to $\overline{\mathbb{R}}$. If for every set n such that $n \in \text{dom } F$ holds $F(n) \leq 0_{\overline{\mathbb{R}}}$, then F is non-positive.

Let R be a binary relation. We say that R is without $-\infty$ if and only if:

- (Def. 3) $-\infty \notin \text{rng } R$.

We say that R is without $+\infty$ if and only if:

- (Def. 4) $+\infty \notin \text{rng } R$.

Let X be a non empty set and let f be a partial function from X to $\overline{\mathbb{R}}$. Let us observe that f is without $-\infty$ if and only if:

- (Def. 5) For every set x holds $-\infty < f(x)$.

Let us observe that f is without $+\infty$ if and only if:

- (Def. 6) For every set x holds $f(x) < +\infty$.

Next we state four propositions:

- (16) Let X be a non empty set and f be a partial function from X to $\overline{\mathbb{R}}$. Then for every set x such that $x \in \text{dom } f$ holds $-\infty < f(x)$ if and only if f is without $-\infty$.
- (17) Let X be a non empty set and f be a partial function from X to $\overline{\mathbb{R}}$. Then for every set x such that $x \in \text{dom } f$ holds $f(x) < +\infty$ if and only if f is without $+\infty$.
- (18) Let X be a non empty set and f be a partial function from X to $\overline{\mathbb{R}}$. If f is non-negative, then f is without $-\infty$.
- (19) Let X be a non empty set and f be a partial function from X to $\overline{\mathbb{R}}$. If f is non-positive, then f is without $+\infty$.

Let X be a non empty set. Note that every partial function from X to $\overline{\mathbb{R}}$ which is non-negative is also without $-\infty$ and every partial function from X to $\overline{\mathbb{R}}$ which is non-positive is also without $+\infty$.

The following propositions are true:

- (20) Let X be a non empty set, S be a σ -field of subsets of X , and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S . Then f is without $+\infty$ and without $-\infty$.
- (21) Let X be a non empty set, Y be a set, and f be a partial function from X to $\overline{\mathbb{R}}$. If f is non-negative, then $f|Y$ is non-negative.
- (22) Let X be a non empty set and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose f is without $-\infty$ and g is without $-\infty$. Then $\text{dom}(f + g) = \text{dom } f \cap \text{dom } g$.
- (23) Let X be a non empty set and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose f is without $-\infty$ and g is without $+\infty$. Then $\text{dom}(f - g) = \text{dom } f \cap \text{dom } g$.
- (24) Let X be a non empty set, S be a σ -field of subsets of X , f, g be partial functions from X to $\overline{\mathbb{R}}$, F be a function from $\mathbb{Q}$ into S , r be a real number, and A be an element of S . Suppose f is without $-\infty$ and g is without $-\infty$ and for every rational number p holds $F(p) = A \cap \text{LE-dom}(f, \overline{\mathbb{R}}(p)) \cap (A \cap \text{LE-dom}(g, \overline{\mathbb{R}}(r - p)))$. Then $A \cap \text{LE-dom}(f + g, \overline{\mathbb{R}}(r)) = \bigcup \text{rng } F$.

Let X be a non empty set and let f be a partial function from X to $\mathbb{R}$. The functor $\overline{\mathbb{R}}(f)$ yielding a partial function from X to $\overline{\mathbb{R}}$ is defined as follows:

(Def. 7) $\overline{\mathbb{R}}(f) = f$.

Next we state a number of propositions:

- (25) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , and f, g be partial functions from X to $\overline{\mathbb{R}}$. If f is non-negative and g is non-negative, then $f + g$ is non-negative.
- (26) Let X be a non empty set, f be a partial function from X to $\overline{\mathbb{R}}$, and c be a real number such that f is non-negative. Then
 - (i) if $0 \leq c$, then cf is non-negative, and

- (ii) if $c \leq 0$, then $c f$ is non-positive.
- (27) Let X be a non empty set and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that for every set x such that $x \in \text{dom } f \cap \text{dom } g$ holds $g(x) \leq f(x)$ and $-\infty < g(x)$ and $f(x) < +\infty$. Then $f - g$ is non-negative.
- (28) Let X be a non empty set and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose f is non-negative and g is non-negative. Then $\text{dom}(f + g) = \text{dom } f \cap \text{dom } g$ and $f + g$ is non-negative.
- (29) Let X be a non empty set and f, g, h be partial functions from X to $\overline{\mathbb{R}}$. Suppose f is non-negative and g is non-negative and h is non-negative. Then $\text{dom}(f + g + h) = \text{dom } f \cap \text{dom } g \cap \text{dom } h$ and $f + g + h$ is non-negative and for every set x such that $x \in \text{dom } f \cap \text{dom } g \cap \text{dom } h$ holds $(f + g + h)(x) = f(x) + g(x) + h(x)$.
- (30) Let X be a non empty set and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose f is without $-\infty$ and g is without $-\infty$. Then
 - (i) $\text{dom}(\max_+(f + g) + \max_-(f)) = \text{dom } f \cap \text{dom } g$,
 - (ii) $\text{dom}(\max_-(f + g) + \max_+(f)) = \text{dom } f \cap \text{dom } g$,
 - (iii) $\text{dom}(\max_+(f + g) + \max_-(f) + \max_-(g)) = \text{dom } f \cap \text{dom } g$,
 - (iv) $\text{dom}(\max_-(f + g) + \max_+(f) + \max_+(g)) = \text{dom } f \cap \text{dom } g$,
 - (v) $\max_+(f + g) + \max_-(f)$ is non-negative, and
 - (vi) $\max_-(f + g) + \max_+(f)$ is non-negative.
- (31) Let X be a non empty set and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose f is without $-\infty$ and without $+\infty$ and g is without $-\infty$ and without $+\infty$. Then $\max_+(f + g) + \max_-(f) + \max_-(g) = \max_-(f + g) + \max_+(f) + \max_+(g)$.
- (32) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and c be a real number. If $0 \leq c$, then $\max_+(c f) = c \max_+(f)$ and $\max_-(c f) = c \max_-(f)$.
- (33) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and c be a real number. If $0 \leq c$, then $\max_+((-c) f) = c \max_-(f)$ and $\max_-((-c) f) = c \max_+(f)$.
- (34) Let X be a non empty set, f be a partial function from X to $\overline{\mathbb{R}}$, and A be a set. Then $\max_+(f \upharpoonright A) = \max_+(f) \upharpoonright A$ and $\max_-(f \upharpoonright A) = \max_-(f) \upharpoonright A$.
- (35) Let X be a non empty set, f, g be partial functions from X to $\overline{\mathbb{R}}$, and B be a set. If $B \subseteq \text{dom}(f + g)$, then $\text{dom}((f + g) \upharpoonright B) = B$ and $\text{dom}(f \upharpoonright B + g \upharpoonright B) = B$ and $(f + g) \upharpoonright B = f \upharpoonright B + g \upharpoonright B$.
- (36) Let X be a non empty set, f be a partial function from X to $\overline{\mathbb{R}}$, and a be an extended real number. Then $\text{EQ-dom}(f, a) = f^{-1}(\{a\})$.

3. LEMMAS FOR MEASURABLE FUNCTION AND SIMPLE VALUED FUNCTION

The following propositions are true:

- (37) Let X be a non empty set, S be a σ -field of subsets of X , f, g be partial functions from X to $\overline{\mathbb{R}}$, and A be an element of S . Suppose f is without $-\infty$ and g is without $-\infty$ and f is measurable on A and g is measurable on A . Then $f + g$ is measurable on A .
- (38) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S and $\text{dom } f = \emptyset$. Then there exists a finite sequence F of separated subsets of S and there exist finite sequences a, x of elements of $\overline{\mathbb{R}}$ such that
 - (i) F and a are representation of f ,
 - (ii) $a(1) = 0$,
 - (iii) for every natural number n such that $2 \leq n$ and $n \in \text{dom } a$ holds $0 < a(n)$ and $a(n) < +\infty$,
 - (iv) $\text{dom } x = \text{dom } F$,
 - (v) for every natural number n such that $n \in \text{dom } x$ holds $x(n) = a(n) \cdot (M \cdot F)(n)$, and
 - (vi) $\sum x = 0$.
- (39) Let X be a non empty set, S be a σ -field of subsets of X , f be a partial function from X to $\overline{\mathbb{R}}$, A be an element of S , and r, s be real numbers. Suppose f is measurable on A and $A \subseteq \text{dom } f$. Then $A \cap \text{GTE-dom}(f, \overline{\mathbb{R}}(r)) \cap \text{LE-dom}(f, \overline{\mathbb{R}}(s))$ is measurable on S .
- (40) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , f be a partial function from X to $\overline{\mathbb{R}}$, and A be an element of S . If f is simple function in S , then $f|A$ is simple function in S .
- (41) Let X be a non empty set, S be a σ -field of subsets of X , A be an element of S , F be a finite sequence of separated subsets of S , and G be a finite sequence. Suppose $\text{dom } F = \text{dom } G$ and for every natural number n such that $n \in \text{dom } F$ holds $G(n) = F(n) \cap A$. Then G is a finite sequence of separated subsets of S .
- (42) Let X be a non empty set, S be a σ -field of subsets of X , f be a partial function from X to $\overline{\mathbb{R}}$, A be an element of S , F, G be finite sequences of separated subsets of S , and a be a finite sequence of elements of $\overline{\mathbb{R}}$. Suppose $\text{dom } F = \text{dom } G$ and for every natural number n such that $n \in \text{dom } F$ holds $G(n) = F(n) \cap A$ and F and a are representation of f . Then G and a are representation of $f|A$.
- (43) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , and f be a partial function from X to $\overline{\mathbb{R}}$. If f is simple function in S , then $\text{dom } f$ is an element of S .

- (44) Let X be a non empty set, S be a σ -field of subsets of X , and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S and g is simple function in S . Then $f + g$ is simple function in S .
- (45) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , f be a partial function from X to $\overline{\mathbb{R}}$, and c be a real number. If f is simple function in S , then cf is simple function in S .
- (46) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that
- (i) f is simple function in S ,
 - (ii) g is simple function in S , and
 - (iii) for every set x such that $x \in \text{dom}(f - g)$ holds $g(x) \leq f(x)$.
- Then $f - g$ is non-negative.
- (47) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , A be an element of S , and c be an extended real number. Suppose $c \neq +\infty$ and $c \neq -\infty$. Then there exists a partial function f from X to $\overline{\mathbb{R}}$ such that f is simple function in S and $\text{dom } f = A$ and for every set x such that $x \in A$ holds $f(x) = c$.
- (48) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , f be a partial function from X to $\overline{\mathbb{R}}$, and B, B_1 be elements of S . Suppose f is measurable on B and $B_1 = \text{dom } f \cap B$. Then $f|_B$ is measurable on B_1 .
- (49) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , A be an element of S , and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that
- (i) $A \subseteq \text{dom } f$,
 - (ii) f is measurable on A ,
 - (iii) g is measurable on A ,
 - (iv) f is without $-\infty$, and
 - (v) g is without $-\infty$.
- Then $\max_+(f + g) + \max_-(f)$ is measurable on A .
- (50) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , A be an element of S , and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that
- (i) $A \subseteq \text{dom } f \cap \text{dom } g$,
 - (ii) f is measurable on A ,
 - (iii) g is measurable on A ,
 - (iv) f is without $-\infty$, and
 - (v) g is without $-\infty$.
- Then $\max_-(f + g) + \max_+(f)$ is measurable on A .
- (51) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , and A be a set. If $A \in S$, then $0 \leq M(A)$.

- (52) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that
- (i) there exists an element E_1 of S such that $E_1 = \text{dom } f$ and f is measurable on E_1 ,
 - (ii) there exists an element E_2 of S such that $E_2 = \text{dom } g$ and g is measurable on E_2 ,
 - (iii) $f^{-1}(\{+\infty\}) \in S$,
 - (iv) $f^{-1}(\{-\infty\}) \in S$,
 - (v) $g^{-1}(\{+\infty\}) \in S$, and
 - (vi) $g^{-1}(\{-\infty\}) \in S$.
- Then $\text{dom}(f + g) \in S$.
- (53) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that
- (i) there exists an element E_1 of S such that $E_1 = \text{dom } f$ and f is measurable on E_1 , and
 - (ii) there exists an element E_2 of S such that $E_2 = \text{dom } g$ and g is measurable on E_2 .
- Then there exists an element E of S such that $E = \text{dom}(f + g)$ and $f + g$ is measurable on E .
- (54) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , f be a partial function from X to $\overline{\mathbb{R}}$, and A, B be elements of S . Suppose $\text{dom } f = A$. Then f is measurable on B if and only if f is measurable on $A \cap B$.
- (55) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , and f be a partial function from X to $\overline{\mathbb{R}}$. Given an element A of S such that $\text{dom } f = A$. Let c be a real number and B be an element of S . If f is measurable on B , then cf is measurable on B .

4. SEQUENCE OF EXTENDED REAL NUMBERS

A sequence of extended reals is a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$.

Let s_1 be a sequence of extended reals. We say that s_1 is convergent to finite number if and only if the condition (Def. 8) is satisfied.

- (Def. 8) There exists a real number g such that for every real number p if $0 < p$, then there exists a natural number n such that for every natural number m such that $n \leq m$ holds $|s_1(m) - \overline{\mathbb{R}}(g)| < p$.

Let s_1 be a sequence of extended reals. We say that s_1 is convergent to $+\infty$ if and only if the condition (Def. 9) is satisfied.

- (Def. 9) Let g be a real number. Suppose $0 < g$. Then there exists a natural number n such that for every natural number m such that $n \leq m$ holds $g \leq s_1(m)$.

Let s_1 be a sequence of extended reals. We say that s_1 is convergent to $-\infty$ if and only if the condition (Def. 10) is satisfied.

- (Def. 10) Let g be a real number. Suppose $g < 0$. Then there exists a natural number n such that for every natural number m such that $n \leq m$ holds $s_1(m) \leq g$.

We now state two propositions:

- (56) Let s_1 be a sequence of extended reals. Suppose s_1 is convergent to $+\infty$. Then s_1 is not convergent to $-\infty$ and s_1 is not convergent to finite number.
- (57) Let s_1 be a sequence of extended reals. Suppose s_1 is convergent to $-\infty$. Then s_1 is not convergent to $+\infty$ and s_1 is not convergent to finite number.

Let s_1 be a sequence of extended reals. We say that s_1 is convergent if and only if:

- (Def. 11) s_1 is convergent to finite number, or convergent to $+\infty$, or convergent to $-\infty$.

Let s_1 be a sequence of extended reals. Let us assume that s_1 is convergent. The functor $\lim s_1$ yields an extended real number and is defined by the conditions (Def. 12).

- (Def. 12)(i) There exists a real number g such that $\lim s_1 = g$ and for every real number p such that $0 < p$ there exists a natural number n such that for every natural number m such that $n \leq m$ holds $|s_1(m) - \lim s_1| < p$ and s_1 is convergent to finite number, or
- (ii) $\lim s_1 = +\infty$ and s_1 is convergent to $+\infty$, or
- (iii) $\lim s_1 = -\infty$ and s_1 is convergent to $-\infty$.

We now state a number of propositions:

- (58) Let s_1 be a sequence of extended reals and r be a real number. Suppose that for every natural number n holds $s_1(n) = r$. Then s_1 is convergent to finite number and $\lim s_1 = r$.
- (59) Let F be a finite sequence of elements of $\overline{\mathbb{R}}$. If for every natural number n such that $n \in \text{dom } F$ holds $0 \leq F(n)$, then $0 \leq \sum F$.
- (60) Let L be a sequence of extended reals. Suppose that for all natural numbers n, m such that $n \leq m$ holds $L(n) \leq L(m)$. Then L is convergent and $\lim L = \sup \text{rng } L$.
- (61) For all sequences L, G of extended reals such that for every natural number n holds $L(n) \leq G(n)$ holds $\sup \text{rng } L \leq \sup \text{rng } G$.
- (62) For every sequence L of extended reals and for every natural number n holds $L(n) \leq \sup \text{rng } L$.
- (63) Let L be a sequence of extended reals and K be an extended real number. If for every natural number n holds $L(n) \leq K$, then $\sup \text{rng } L \leq K$.

- (64) Let L be a sequence of extended reals and K be an extended real number. If $K \neq +\infty$ and for every natural number n holds $L(n) \leq K$, then $\sup \text{rng } L < +\infty$.
- (65) Let L be a sequence of extended reals. Suppose L is without $-\infty$. Then $\sup \text{rng } L \neq +\infty$ if and only if there exists a real number K such that $0 < K$ and for every natural number n holds $L(n) \leq K$.
- (66) Let L be a sequence of extended reals and c be an extended real number. Suppose that for every natural number n holds $L(n) = c$. Then L is convergent and $\lim L = c$ and $\lim L = \sup \text{rng } L$.
- (67) Let J, K, L be sequences of extended reals. Suppose that
- (i) for all natural numbers n, m such that $n \leq m$ holds $J(n) \leq J(m)$,
 - (ii) for all natural numbers n, m such that $n \leq m$ holds $K(n) \leq K(m)$,
 - (iii) J is without $-\infty$,
 - (iv) K is without $-\infty$, and
 - (v) for every natural number n holds $J(n) + K(n) = L(n)$.
- Then L is convergent and $\lim L = \sup \text{rng } L$ and $\lim L = \lim J + \lim K$ and $\sup \text{rng } L = \sup \text{rng } K + \sup \text{rng } J$.
- (68) Let L, K be sequences of extended reals and c be a real number. Suppose $0 \leq c$ and L is without $-\infty$ and for every natural number n holds $K(n) = \overline{\mathbb{R}}(c) \cdot L(n)$. Then $\sup \text{rng } K = \overline{\mathbb{R}}(c) \cdot \sup \text{rng } L$ and K is without $-\infty$.
- (69) Let L, K be sequences of extended reals and c be a real number. Suppose that
- (i) $0 \leq c$,
 - (ii) for all natural numbers n, m such that $n \leq m$ holds $L(n) \leq L(m)$,
 - (iii) for every natural number n holds $K(n) = \overline{\mathbb{R}}(c) \cdot L(n)$, and
 - (iv) L is without $-\infty$.
- Then
- (v) for all natural numbers n, m such that $n \leq m$ holds $K(n) \leq K(m)$,
 - (vi) K is without $-\infty$ and convergent,
 - (vii) $\lim K = \sup \text{rng } K$, and
 - (viii) $\lim K = \overline{\mathbb{R}}(c) \cdot \lim L$.

5. SEQUENCE OF EXTENDED REAL VALUED FUNCTIONS

Let X be a non empty set, let H be a sequence of partial functions from X into $\overline{\mathbb{R}}$, and let x be an element of X . The functor $H\#x$ yields a sequence of extended reals and is defined as follows:

(Def. 13) For every natural number n holds $(H\#x)(n) = H(n)(x)$.

Let D_1, D_2 be sets, let F be a function from $\mathbb{N}$ into $D_1 \rightarrow D_2$, and let n be a natural number. Then $F(n)$ is a partial function from D_1 to D_2 .

Next we state the proposition

- (70) Let X be a non empty set, S be a σ -field of subsets of X , and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose there exists an element A of S such that $A = \text{dom } f$ and f is measurable on A and f is non-negative. Then there exists a sequence F of partial functions from X into $\overline{\mathbb{R}}$ such that
- (i) for every natural number n holds $F(n)$ is simple function in S and $\text{dom } F(n) = \text{dom } f$,
 - (ii) for every natural number n holds $F(n)$ is non-negative,
 - (iii) for all natural numbers n, m such that $n \leq m$ and for every element x of X such that $x \in \text{dom } f$ holds $F(n)(x) \leq F(m)(x)$, and
 - (iv) for every element x of X such that $x \in \text{dom } f$ holds $F\#x$ is convergent and $\lim(F\#x) = f(x)$.

6. INTEGRAL OF NON NEGATIVE SIMPLE VALUED FUNCTION

Let X be a non empty set, let S be a σ -field of subsets of X , let M be a σ -measure on S , and let f be a partial function from X to $\overline{\mathbb{R}}$. The functor $\int' f \, dM$ yielding an element of $\overline{\mathbb{R}}$ is defined as follows:

$$(\text{Def. 14}) \quad \int' f \, dM = \begin{cases} \int_X f \, dM, & \text{if } \text{dom } f \neq \emptyset, \\ 0_{\overline{\mathbb{R}}}, & \text{otherwise.} \end{cases}$$

The following propositions are true:

- (71) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S and g is simple function in S and f is non-negative and g is non-negative. Then $\text{dom}(f + g) = \text{dom } f \cap \text{dom } g$ and $\int' f + g \, dM = \int' f \upharpoonright \text{dom}(f + g) \, dM + \int' g \upharpoonright \text{dom}(f + g) \, dM$.
- (72) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , f be a partial function from X to $\overline{\mathbb{R}}$, and c be a real number. Suppose f is simple function in S and f is non-negative and $0 \leq c$. Then $\int' c f \, dM = \overline{\mathbb{R}}(c) \cdot \int' f \, dM$.
- (73) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , f be a partial function from X to $\overline{\mathbb{R}}$, and A, B be elements of S . Suppose f is simple function in S and f is non-negative and A misses B . Then $\int' f \upharpoonright (A \cup B) \, dM = \int' f \upharpoonright A \, dM + \int' f \upharpoonright B \, dM$.
- (74) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , and f be a partial function from X to $\overline{\mathbb{R}}$. If f is simple function in S and f is non-negative, then $0 \leq \int' f \, dM$.
- (75) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that
 - (i) f is simple function in S ,

- (ii) f is non-negative,
- (iii) g is simple function in S ,
- (iv) g is non-negative, and
- (v) for every set x such that $x \in \text{dom}(f - g)$ holds $g(x) \leq f(x)$.
 Then $\text{dom}(f - g) = \text{dom } f \cap \text{dom } g$ and $\int' f \upharpoonright \text{dom}(f - g) dM = \int' f - g dM + \int' g \upharpoonright \text{dom}(f - g) dM$.
- (76) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that
 - (i) f is simple function in S ,
 - (ii) g is simple function in S ,
 - (iii) f is non-negative,
 - (iv) g is non-negative, and
 - (v) for every set x such that $x \in \text{dom}(f - g)$ holds $g(x) \leq f(x)$.
 Then $\int' g \upharpoonright \text{dom}(f - g) dM \leq \int' f \upharpoonright \text{dom}(f - g) dM$.
- (77) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , f be a partial function from X to $\overline{\mathbb{R}}$, and c be an extended real number. Suppose $0 \leq c$ and f is simple function in S and for every set x such that $x \in \text{dom } f$ holds $f(x) = c$. Then $\int' f dM = c \cdot M(\text{dom } f)$.
- (78) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S and f is non-negative. Then $\int' f \upharpoonright \text{EQ-dom}(f, \overline{\mathbb{R}}(0)) dM = 0$.
- (79) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , B be an element of S , and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S and $M(B) = 0$ and f is non-negative. Then $\int' f \upharpoonright B dM = 0$.
- (80) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , g be a partial function from X to $\overline{\mathbb{R}}$, F be a sequence of partial functions from X into $\overline{\mathbb{R}}$, and L be a sequence of extended reals. Suppose that g is simple function in S and for every set x such that $x \in \text{dom } g$ holds $0 < g(x)$ and for every natural number n holds $F(n)$ is simple function in S and for every natural number n holds $\text{dom } F(n) = \text{dom } g$ and for every natural number n holds $F(n)$ is non-negative and for all natural numbers n, m such that $n \leq m$ and for every element x of X such that $x \in \text{dom } g$ holds $F(n)(x) \leq F(m)(x)$ and for every element x of X such that $x \in \text{dom } g$ holds $F \# x$ is convergent and $g(x) \leq \lim(F \# x)$ and for every natural number n holds $L(n) = \int' F(n) dM$. Then L is convergent and $\int' g dM \leq \lim L$.
- (81) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , g be a partial function from X to $\overline{\mathbb{R}}$, and F be a sequence of partial functions from X into $\overline{\mathbb{R}}$. Suppose that g is simple function in S and g is non-negative and for every natural number n holds $F(n)$ is simple

function in S and for every natural number n holds $\text{dom } F(n) = \text{dom } g$ and for every natural number n holds $F(n)$ is non-negative and for all natural numbers n, m such that $n \leq m$ and for every element x of X such that $x \in \text{dom } g$ holds $F(n)(x) \leq F(m)(x)$ and for every element x of X such that $x \in \text{dom } g$ holds $F \# x$ is convergent and $g(x) \leq \lim(F \# x)$. Then there exists a sequence G of extended reals such that for every natural number n holds $G(n) = \int' F(n) dM$ and G is convergent and $\sup \text{rng } G = \lim G$ and $\int' g dM \leq \lim G$.

- (82) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , A be an element of S , F, G be sequences of partial functions from X into $\overline{\mathbb{R}}$, and K, L be sequences of extended reals. Suppose that for every natural number n holds $F(n)$ is simple function in S and $\text{dom } F(n) = A$ and for every natural number n holds $F(n)$ is non-negative and for all natural numbers n, m such that $n \leq m$ and for every element x of X such that $x \in A$ holds $F(n)(x) \leq F(m)(x)$ and for every natural number n holds $G(n)$ is simple function in S and $\text{dom } G(n) = A$ and for every natural number n holds $G(n)$ is non-negative and for all natural numbers n, m such that $n \leq m$ and for every element x of X such that $x \in A$ holds $G(n)(x) \leq G(m)(x)$ and for every element x of X such that $x \in A$ holds $F \# x$ is convergent and $G \# x$ is convergent and $\lim(F \# x) = \lim(G \# x)$ and for every natural number n holds $K(n) = \int' F(n) dM$ and $L(n) = \int' G(n) dM$. Then K is convergent and L is convergent and $\lim K = \lim L$.

Let X be a non empty set, let S be a σ -field of subsets of X , let M be a σ -measure on S , and let f be a partial function from X to $\overline{\mathbb{R}}$. Let us assume that there exists an element A of S such that $A = \text{dom } f$ and f is measurable on A and f is non-negative. The functor $\int^+ f dM$ yielding an element of $\overline{\mathbb{R}}$ is defined by the condition (Def. 15).

- (Def. 15) There exists a sequence F of partial functions from X into $\overline{\mathbb{R}}$ and there exists a sequence K of extended reals such that
 for every natural number n holds $F(n)$ is simple function in S and $\text{dom } F(n) = \text{dom } f$ and for every natural number n holds $F(n)$ is non-negative and for all natural numbers n, m such that $n \leq m$ and for every element x of X such that $x \in \text{dom } f$ holds $F(n)(x) \leq F(m)(x)$ and for every element x of X such that $x \in \text{dom } f$ holds $F \# x$ is convergent and $\lim(F \# x) = f(x)$ and for every natural number n holds $K(n) = \int' F(n) dM$ and K is convergent and $\int^+ f dM = \lim K$.

The following propositions are true:

- (83) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , and f be a partial function from X to $\overline{\mathbb{R}}$. If f is simple function in S and f is non-negative, then $\int^+ f dM = \int' f dM$.
- (84) Let X be a non empty set, S be a σ -field of subsets of X , M be a

- σ -measure on S , and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that
- (i) there exists an element A of S such that $A = \text{dom } f$ and f is measurable on A ,
 - (ii) there exists an element B of S such that $B = \text{dom } g$ and g is measurable on B ,
 - (iii) f is non-negative, and
 - (iv) g is non-negative.
- Then there exists an element C of S such that $C = \text{dom}(f + g)$ and $\int^+ f + g \, dM = \int^+ f \, dM + \int^+ g \, dM$.
- (85) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose there exists an element A of S such that $A = \text{dom } f$ and f is measurable on A and f is non-negative. Then $0 \leq \int^+ f \, dM$.
 - (86) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , f be a partial function from X to $\overline{\mathbb{R}}$, and A be an element of S . Suppose there exists an element E of S such that $E = \text{dom } f$ and f is measurable on E and f is non-negative. Then $0 \leq \int^+ f \, dM$.
 - (87) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , f be a partial function from X to $\overline{\mathbb{R}}$, and A, B be elements of S . Suppose there exists an element E of S such that $E = \text{dom } f$ and f is measurable on E and f is non-negative and A misses B . Then $\int^+ f \, dM = \int^+ f \, dM$.
 - (88) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , f be a partial function from X to $\overline{\mathbb{R}}$, and A be an element of S . Suppose there exists an element E of S such that $E = \text{dom } f$ and f is measurable on E and f is non-negative and $M(A) = 0$. Then $\int^+ f \, dM = 0$.
 - (89) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , f be a partial function from X to $\overline{\mathbb{R}}$, and A, B be elements of S . Suppose there exists an element E of S such that $E = \text{dom } f$ and f is measurable on E and f is non-negative and $A \subseteq B$. Then $\int^+ f \, dM \leq \int^+ f \, dM$.
 - (90) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , f be a partial function from X to $\overline{\mathbb{R}}$, and E, A be elements of S . Suppose f is non-negative and $E = \text{dom } f$ and f is measurable on E and $M(A) = 0$. Then $\int^+ f \, dM = \int^+ f \, dM$.
 - (91) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that
 - (i) there exists an element E of S such that $E = \text{dom } f$ and $E = \text{dom } g$ and f is measurable on E and g is measurable on E ,
 - (ii) f is non-negative,

- (iii) g is non-negative, and
- (iv) for every element x of X such that $x \in \text{dom } g$ holds $g(x) \leq f(x)$.
Then $\int^+ g \, dM \leq \int^+ f \, dM$.
- (92) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , f be a partial function from X to $\overline{\mathbb{R}}$, and c be a real number. Suppose $0 \leq c$ and there exists an element A of S such that $A = \text{dom } f$ and f is measurable on A and f is non-negative. Then $\int^+ c f \, dM = \overline{\mathbb{R}}(c) \cdot \int^+ f \, dM$.
- (93) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose that
 - (i) there exists an element A of S such that $A = \text{dom } f$ and f is measurable on A , and
 - (ii) for every element x of X such that $x \in \text{dom } f$ holds $0 = f(x)$.
Then $\int^+ f \, dM = 0$.

7. INTEGRAL OF MEASURABLE FUNCTION

Let X be a non empty set, let S be a σ -field of subsets of X , let M be a σ -measure on S , and let f be a partial function from X to $\overline{\mathbb{R}}$. The functor $\int f \, dM$ yielding an element of $\overline{\mathbb{R}}$ is defined as follows:

$$(\text{Def. 16}) \quad \int f \, dM = \int^+ \max_+(f) \, dM - \int^+ \max_-(f) \, dM.$$

We now state several propositions:

- (94) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose there exists an element A of S such that $A = \text{dom } f$ and f is measurable on A and f is non-negative. Then $\int f \, dM = \int^+ f \, dM$.
- (95) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S and f is non-negative. Then $\int f \, dM = \int^+ f \, dM$ and $\int f \, dM = \int' f \, dM$.
- (96) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose there exists an element A of S such that $A = \text{dom } f$ and f is measurable on A and f is non-negative. Then $0 \leq \int f \, dM$.
- (97) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , f be a partial function from X to $\overline{\mathbb{R}}$, and A, B be elements of S . Suppose there exists an element E of S such that $E = \text{dom } f$ and f is measurable on E and f is non-negative and A misses B . Then $\int f \upharpoonright (A \cup B) \, dM = \int f \upharpoonright A \, dM + \int f \upharpoonright B \, dM$.

- (98) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , f be a partial function from X to $\overline{\mathbb{R}}$, and A be an element of S . Suppose there exists an element E of S such that $E = \text{dom } f$ and f is measurable on E and f is non-negative. Then $0 \leq \int f \upharpoonright A \, dM$.
- (99) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , f be a partial function from X to $\overline{\mathbb{R}}$, and A, B be elements of S . Suppose there exists an element E of S such that $E = \text{dom } f$ and f is measurable on E and f is non-negative and $A \subseteq B$. Then $\int f \upharpoonright A \, dM \leq \int f \upharpoonright B \, dM$.
- (100) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , f be a partial function from X to $\overline{\mathbb{R}}$, and A be an element of S . Suppose there exists an element E of S such that $E = \text{dom } f$ and f is measurable on E and $M(A) = 0$. Then $\int f \upharpoonright A \, dM = 0$.
- (101) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , f be a partial function from X to $\overline{\mathbb{R}}$, and E, A be elements of S . If $E = \text{dom } f$ and f is measurable on E and $M(A) = 0$, then $\int f \upharpoonright (E \setminus A) \, dM = \int f \, dM$.

Let X be a non empty set, let S be a σ -field of subsets of X , let M be a σ -measure on S , and let f be a partial function from X to $\overline{\mathbb{R}}$. We say that f is integrable on M if and only if:

- (Def. 17) There exists an element A of S such that $A = \text{dom } f$ and f is measurable on A and $\int^+ \max_+(f) \, dM < +\infty$ and $\int^+ \max_-(f) \, dM < +\infty$.

One can prove the following propositions:

- (102) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose f is integrable on M . Then $0 \leq \int^+ \max_+(f) \, dM$ and $0 \leq \int^+ \max_-(f) \, dM$ and $-\infty < \int f \, dM$ and $\int f \, dM < +\infty$.
- (103) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , f be a partial function from X to $\overline{\mathbb{R}}$, and A be an element of S . Suppose f is integrable on M . Then $\int^+ \max_+(f \upharpoonright A) \, dM \leq \int^+ \max_+(f) \, dM$ and $\int^+ \max_-(f \upharpoonright A) \, dM \leq \int^+ \max_-(f) \, dM$ and $f \upharpoonright A$ is integrable on M .
- (104) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , f be a partial function from X to $\overline{\mathbb{R}}$, and A, B be elements of S . Suppose f is integrable on M and A misses B . Then $\int f \upharpoonright (A \cup B) \, dM = \int f \upharpoonright A \, dM + \int f \upharpoonright B \, dM$.
- (105) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , f be a partial function from X to $\overline{\mathbb{R}}$, and A, B be elements of S . Suppose f is integrable on M and $B = \text{dom } f \setminus A$. Then $f \upharpoonright A$ is integrable on M and $\int f \, dM = \int f \upharpoonright A \, dM + \int f \upharpoonright B \, dM$.

- (106) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , and f be a partial function from X to $\overline{\mathbb{R}}$. Given an element A of S such that $A = \text{dom } f$ and f is measurable on A . Then f is integrable on M if and only if $|f|$ is integrable on M .
- (107) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , and f be a partial function from X to $\overline{\mathbb{R}}$. If f is integrable on M , then $|\int f \, dM| \leq \int |f| \, dM$.
- (108) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that
- (i) there exists an element A of S such that $A = \text{dom } f$ and f is measurable on A ,
 - (ii) $\text{dom } f = \text{dom } g$,
 - (iii) g is integrable on M , and
 - (iv) for every element x of X such that $x \in \text{dom } f$ holds $|f(x)| \leq g(x)$.
- Then f is integrable on M and $\int |f| \, dM \leq \int g \, dM$.
- (109) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , f be a partial function from X to $\overline{\mathbb{R}}$, and r be a real number. Suppose $\text{dom } f \in S$ and $0 \leq r$ and $\text{dom } f \neq \emptyset$ and for every set x such that $x \in \text{dom } f$ holds $f(x) = r$. Then $\int_X f \, dM = \overline{\mathbb{R}}(r) \cdot M(\text{dom } f)$.
- (110) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , f be a partial function from X to $\overline{\mathbb{R}}$, and r be a real number. Suppose $\text{dom } f \in S$ and $0 \leq r$ and for every set x such that $x \in \text{dom } f$ holds $f(x) = r$. Then $\int' f \, dM = \overline{\mathbb{R}}(r) \cdot M(\text{dom } f)$.
- (111) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose f is integrable on M . Then $f^{-1}(\{+\infty\}) \in S$ and $f^{-1}(\{-\infty\}) \in S$ and $M(f^{-1}(\{+\infty\})) = 0$ and $M(f^{-1}(\{-\infty\})) = 0$ and $f^{-1}(\{+\infty\}) \cup f^{-1}(\{-\infty\}) \in S$ and $M(f^{-1}(\{+\infty\}) \cup f^{-1}(\{-\infty\})) = 0$.
- (112) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose f is integrable on M and g is integrable on M and f is non-negative and g is non-negative. Then $f + g$ is integrable on M .
- (113) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , and f, g be partial functions from X to $\overline{\mathbb{R}}$. If f is integrable on M and g is integrable on M , then $\text{dom}(f + g) \in S$.
- (114) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose f is integrable on M and g is integrable on M . Then $f + g$ is integrable on M .
- (115) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose f is

- integrable on M and g is integrable on M . Then there exists an element E of S such that $E = \text{dom } f \cap \text{dom } g$ and $\int f+g \, dM = \int f \upharpoonright E \, dM + \int g \upharpoonright E \, dM$.
- (116) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , f be a partial function from X to $\overline{\mathbb{R}}$, and c be a real number. Suppose f is integrable on M . Then $c f$ is integrable on M and $\int c f \, dM = \overline{\mathbb{R}}(c) \cdot \int f \, dM$.

Let X be a non empty set, let S be a σ -field of subsets of X , let M be a σ -measure on S , let f be a partial function from X to $\overline{\mathbb{R}}$, and let B be an element of S . The functor $\int_B f \, dM$ yielding an element of $\overline{\mathbb{R}}$ is defined as follows:

$$\text{(Def. 18)} \quad \int_B f \, dM = \int f \upharpoonright B \, dM.$$

The following propositions are true:

- (117) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , f, g be partial functions from X to $\overline{\mathbb{R}}$, and B be an element of S . Suppose f is integrable on M and g is integrable on M and $B \subseteq \text{dom}(f+g)$. Then $f+g$ is integrable on M and $\int_B f+g \, dM = \int_B f \, dM + \int_B g \, dM$.
- (118) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , f be a partial function from X to $\overline{\mathbb{R}}$, c be a real number, and B be an element of S . Suppose f is integrable on M and f is measurable on B . Then $f \upharpoonright B$ is integrable on M and $\int_B c f \, dM = \overline{\mathbb{R}}(c) \cdot \int_B f \, dM$.

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