

Partial Differentiation on Normed Linear Spaces $\mathcal{R}^n$

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Summary. In this article, we define the partial differentiation of functions of real variable and prove the linearity of this operator [18].

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The notation and terminology used here are introduced in the following papers: [21], [24], [25], [5], [26], [7], [6], [15], [13], [3], [1], [20], [11], [22], [23], [14], [8], [2], [4], [27], [28], [16], [9], [19], [17], [12], and [10].

1. PRELIMINARIES

Let i, n be elements of $\mathbb{N}$. The functor $\text{proj}(i, n)$ yielding a function from $\mathcal{R}^n$ into $\mathbb{R}$ is defined by:

(Def. 1) For every element x of $\mathcal{R}^n$ holds $(\text{proj}(i, n))(x) = x(i)$.

Next we state two propositions:

- (1) $\text{dom proj}(1, 1) = \mathcal{R}^1$ and $\text{rng proj}(1, 1) = \mathbb{R}$ and for every element x of $\mathbb{R}$ holds $(\text{proj}(1, 1))(\langle x \rangle) = x$ and $(\text{proj}(1, 1))^{-1}(x) = \langle x \rangle$.
- (2)(i) $(\text{proj}(1, 1))^{-1}$ is a function from $\mathbb{R}$ into $\mathcal{R}^1$,
- (ii) $(\text{proj}(1, 1))^{-1}$ is one-to-one,
- (iii) $\text{dom}((\text{proj}(1, 1))^{-1}) = \mathbb{R}$,
- (iv) $\text{rng}((\text{proj}(1, 1))^{-1}) = \mathcal{R}^1$, and

- (v) there exists a function g from $\mathbb{R}$ into $\mathcal{R}^1$ such that g is bijective and $(\text{proj}(1, 1))^{-1} = g$.

One can check that $\text{proj}(1, 1)$ is bijective.

Let g be a partial function from $\mathbb{R}$ to $\mathbb{R}$. The functor $\langle g \rangle$ yields a partial function from $\mathcal{R}^1$ to $\mathcal{R}^1$ and is defined as follows:

- (Def. 2) $\langle g \rangle = (\text{proj}(1, 1))^{-1} \cdot g \cdot \text{proj}(1, 1)$.

Let n be an element of $\mathbb{N}$ and let g be a partial function from $\mathcal{R}^n$ to $\mathbb{R}$. The functor $\langle g \rangle$ yielding a partial function from $\mathcal{R}^n$ to $\mathcal{R}^1$ is defined as follows:

- (Def. 3) $\langle g \rangle = (\text{proj}(1, 1))^{-1} \cdot g$.

Let i, n be elements of $\mathbb{N}$. The functor $\text{Proj}(i, n)$ yielding a function from $\langle \mathcal{E}^n, \|\cdot\| \rangle$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$ is defined as follows:

- (Def. 4) For every point x of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ holds $(\text{Proj}(i, n))(x) = \langle (\text{proj}(i, n))(x) \rangle$.

Let i be an element of $\mathbb{N}$ and let x be a finite sequence of elements of $\mathbb{R}$. The functor $\text{reproj}(i, x)$ yielding a function is defined as follows:

- (Def. 5) $\text{dom reproj}(i, x) = \mathbb{R}$ and for every element r of $\mathbb{R}$ holds $(\text{reproj}(i, x))(r) = \text{Replace}(x, i, r)$.

Let n, i be elements of $\mathbb{N}$ and let x be an element of $\mathcal{R}^n$. Then $\text{reproj}(i, x)$ is a function from $\mathbb{R}$ into $\mathcal{R}^n$.

Let n, i be elements of $\mathbb{N}$ and let x be a point of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. The functor $\text{reproj}(i, x)$ yielding a function from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$ is defined by the condition (Def. 6).

- (Def. 6) Let r be an element of $\langle \mathcal{E}^1, \|\cdot\| \rangle$. Then there exists an element q of $\mathbb{R}$ and there exists an element y of $\mathcal{R}^n$ such that $r = \langle q \rangle$ and $y = x$ and $(\text{reproj}(i, x))(r) = (\text{reproj}(i, y))(q)$.

Let m, n be non empty elements of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and let x be an element of $\mathcal{R}^m$. We say that f is differentiable in x if and only if the condition (Def. 7) is satisfied.

- (Def. 7) There exists a partial function g from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and there exists a point y of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $f = g$ and $x = y$ and g is differentiable in y .

Let m, n be non empty elements of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and let x be an element of $\mathcal{R}^m$. Let us assume that f is differentiable in x . The functor $f'(x)$ yields a function from $\mathcal{R}^m$ into $\mathcal{R}^n$ and is defined as follows:

- (Def. 8) There exists a partial function g from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and there exists a point y of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $f = g$ and $x = y$ and $f'(x) = g'(y)$.

We now state four propositions:

- (3) Let I be a function from $\mathbb{R}$ into $\mathcal{R}^1$. Suppose $I = (\text{proj}(1, 1))^{-1}$. Then

- (i) for every vector x of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and for every element y of $\mathbb{R}$ such that $x = I(y)$ holds $\|x\| = |y|$,
 - (ii) for all vectors x, y of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and for all elements a, b of $\mathbb{R}$ such that $x = I(a)$ and $y = I(b)$ holds $x + y = I(a + b)$,
 - (iii) for every vector x of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and for every element y of $\mathbb{R}$ and for every real number a such that $x = I(y)$ holds $a \cdot x = I(a \cdot y)$,
 - (iv) for every vector x of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and for every element a of $\mathbb{R}$ such that $x = I(a)$ holds $-x = I(-a)$, and
 - (v) for all vectors x, y of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and for all elements a, b of $\mathbb{R}$ such that $x = I(a)$ and $y = I(b)$ holds $x - y = I(a - b)$.
- (4) Let J be a function from $\mathcal{R}^1$ into $\mathbb{R}$. Suppose $J = \text{proj}(1, 1)$. Then
- (i) for every vector x of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and for every element y of $\mathbb{R}$ such that $J(x) = y$ holds $\|x\| = |y|$,
 - (ii) for all vectors x, y of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and for all elements a, b of $\mathbb{R}$ such that $J(x) = a$ and $J(y) = b$ holds $J(x + y) = a + b$,
 - (iii) for every vector x of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and for every element y of $\mathbb{R}$ and for every real number a such that $J(x) = y$ holds $J(a \cdot x) = a \cdot y$,
 - (iv) for every vector x of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and for every element a of $\mathbb{R}$ such that $J(x) = a$ holds $J(-x) = -a$, and
 - (v) for all vectors x, y of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and for all elements a, b of $\mathbb{R}$ such that $J(x) = a$ and $J(y) = b$ holds $J(x - y) = a - b$.
- (5) Let I be a function from $\mathbb{R}$ into $\mathcal{R}^1$ and J be a function from $\mathcal{R}^1$ into $\mathbb{R}$. Suppose $I = (\text{proj}(1, 1))^{-1}$ and $J = \text{proj}(1, 1)$. Then
- (i) for every rest R of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, $\langle \mathcal{E}^1, \|\cdot\| \rangle$ holds $J \cdot R \cdot I$ is a rest, and
 - (ii) for every linear operator L from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$ holds $J \cdot L \cdot I$ is a linear function.
- (6) Let I be a function from $\mathbb{R}$ into $\mathcal{R}^1$ and J be a function from $\mathcal{R}^1$ into $\mathbb{R}$. Suppose $I = (\text{proj}(1, 1))^{-1}$ and $J = \text{proj}(1, 1)$. Then
- (i) for every rest R holds $I \cdot R \cdot J$ is a rest of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, $\langle \mathcal{E}^1, \|\cdot\| \rangle$, and
 - (ii) for every linear function L holds $I \cdot L \cdot J$ is a bounded linear operator from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$.

In the sequel f is a partial function from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ to $\langle \mathcal{E}^1, \|\cdot\| \rangle$, g is a partial function from $\mathbb{R}$ to $\mathbb{R}$, x is a point of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, and y is an element of $\mathbb{R}$.

We now state four propositions:

- (7) If $f = \langle g \rangle$ and $x = \langle y \rangle$ and f is differentiable in x , then g is differentiable in y and $g'(y) = (\text{proj}(1, 1) \cdot f'(x) \cdot (\text{proj}(1, 1))^{-1})(1)$.
- (8) If $f = \langle g \rangle$ and $x = \langle y \rangle$ and g is differentiable in y , then f is differentiable in x and $f'(x)(\langle 1 \rangle) = \langle g'(y) \rangle$.
- (9) If $f = \langle g \rangle$ and $x = \langle y \rangle$, then f is differentiable in x iff g is differentiable in y .

- (10) If $f = \langle g \rangle$ and $x = \langle y \rangle$ and f is differentiable in x , then $f'(x)(\langle 1 \rangle) = \langle g'(y) \rangle$.

2. PARTIAL DIFFERENTIATION

For simplicity, we adopt the following rules: m, n are non empty elements of $\mathbb{N}$, i, j are elements of $\mathbb{N}$, f is a partial function from $\langle \mathcal{E}^n, \|\cdot\| \rangle$ to $\langle \mathcal{E}^1, \|\cdot\| \rangle$, g is a partial function from $\mathcal{R}^n$ to $\mathbb{R}$, x is a point of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and y is an element of $\mathcal{R}^n$.

Let n, m be non empty elements of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and let x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. We say that f is partially differentiable in x w.r.t. i if and only if:

- (Def. 9) $f \cdot \text{reproj}(i, x)$ is differentiable in $(\text{Proj}(i, m))(x)$.

Let m, n be non empty elements of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and let x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. The functor $\text{partdiff}(f, x, i)$ yielding a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$ is defined as follows:

- (Def. 10) $\text{partdiff}(f, x, i) = (f \cdot \text{reproj}(i, x))'((\text{Proj}(i, m))(x))$.

Let n be a non empty element of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^n$ to $\mathbb{R}$, and let x be an element of $\mathcal{R}^n$. We say that f is partially differentiable in x w.r.t. i if and only if:

- (Def. 11) $f \cdot \text{reproj}(i, x)$ is differentiable in $(\text{proj}(i, n))(x)$.

Let n be a non empty element of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^n$ to $\mathbb{R}$, and let x be an element of $\mathcal{R}^n$. The functor $\text{partdiff}(f, x, i)$ yields a real number and is defined by:

- (Def. 12) $\text{partdiff}(f, x, i) = (f \cdot \text{reproj}(i, x))'((\text{proj}(i, n))(x))$.

We now state several propositions:

- (11) $\text{Proj}(i, n) = (\text{proj}(1, 1))^{-1} \cdot \text{proj}(i, n)$.
- (12) If $x = y$, then $\text{reproj}(i, y) \cdot \text{proj}(1, 1) = \text{reproj}(i, x)$.
- (13) If $f = \langle g \rangle$ and $x = y$, then $\langle g \cdot \text{reproj}(i, y) \rangle = f \cdot \text{reproj}(i, x)$.
- (14) Suppose $f = \langle g \rangle$ and $x = y$. Then f is partially differentiable in x w.r.t. i if and only if g is partially differentiable in y w.r.t. i .
- (15) If $f = \langle g \rangle$ and $x = y$ and f is partially differentiable in x w.r.t. i , then $(\text{partdiff}(f, x, i))(\langle 1 \rangle) = \langle \text{partdiff}(g, y, i) \rangle$.

Let m, n be non empty elements of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and let x be an element of $\mathcal{R}^m$. We say that f is partially differentiable in x w.r.t. i if and only if the condition (Def. 13) is satisfied.

- (Def. 13) There exists a partial function g from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and there exists a point y of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $f = g$ and $x = y$ and g is partially differentiable in y w.r.t. i .

Let m, n be non empty elements of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and let x be an element of $\mathcal{R}^m$. Let us assume that f is partially differentiable in x w.r.t. i . The functor $\text{partdiff}(f, x, i)$ yielding an element of $\mathcal{R}^n$ is defined as follows:

- (Def. 14) There exists a partial function g from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and there exists a point y of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $f = g$ and $x = y$ and $\text{partdiff}(f, x, i) = (\text{partdiff}(g, y, i))(\langle 1 \rangle)$.

One can prove the following four propositions:

- (16) Let m, n be non empty elements of $\mathbb{N}$, F be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, G be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, and y be an element of $\mathcal{R}^m$. Suppose $F = G$ and $x = y$. Then F is partially differentiable in x w.r.t. i if and only if G is partially differentiable in y w.r.t. i .
- (17) Let m, n be non empty elements of $\mathbb{N}$, F be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, G be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, and y be an element of $\mathcal{R}^m$. Suppose $F = G$ and $x = y$ and F is partially differentiable in x w.r.t. i . Then $(\text{partdiff}(F, x, i))(\langle 1 \rangle) = \text{partdiff}(G, y, i)$.
- (18) Let g_1 be a partial function from $\mathcal{R}^n$ to $\mathcal{R}^1$. Suppose $g_1 = \langle g \rangle$. Then g_1 is partially differentiable in y w.r.t. i if and only if g is partially differentiable in y w.r.t. i .
- (19) Let g_1 be a partial function from $\mathcal{R}^n$ to $\mathcal{R}^1$. Suppose $g_1 = \langle g \rangle$ and g_1 is partially differentiable in y w.r.t. i . Then $\text{partdiff}(g_1, y, i) = \langle \text{partdiff}(g, y, i) \rangle$.

3. LINEARITY OF PARTIAL DIFFERENTIAL OPERATOR

For simplicity, we use the following convention: X is a set, r is a real number, f, f_1, f_2 are partial functions from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, g, g_1, g_2 are partial functions from $\mathcal{R}^n$ to $\mathbb{R}$, h is a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, x is a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, y is an element of $\mathcal{R}^n$, and z is an element of $\mathcal{R}^m$.

Let m, n be non empty elements of $\mathbb{N}$, let i, j be elements of $\mathbb{N}$, let f be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and let x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$.

We say that f is partially differentiable in x w.r.t. i and j if and only if:

- (Def. 15) $\text{Proj}(j, n) \cdot f \cdot \text{reproj}(i, x)$ is differentiable in $(\text{Proj}(i, m))(x)$.

Let m, n be non empty elements of $\mathbb{N}$, let i, j be elements of $\mathbb{N}$, let f be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and let x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$.

The functor $\text{partdiff}(f, x, i, j)$ yields a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and is defined by:

$$(\text{Def. 16}) \quad \text{partdiff}(f, x, i, j) = (\text{Proj}(j, n) \cdot f \cdot \text{reproj}(i, x))'((\text{Proj}(i, m))(x)).$$

Let m, n be non empty elements of $\mathbb{N}$, let i, j be elements of $\mathbb{N}$, let h be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and let z be an element of $\mathcal{R}^m$. We say that h is partially differentiable in z w.r.t. i and j if and only if:

$$(\text{Def. 17}) \quad \text{proj}(j, n) \cdot h \cdot \text{reproj}(i, z) \text{ is differentiable in } (\text{proj}(i, m))(z).$$

Let m, n be non empty elements of $\mathbb{N}$, let i, j be elements of $\mathbb{N}$, let h be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and let z be an element of $\mathcal{R}^m$. The functor $\text{partdiff}(h, z, i, j)$ yielding a real number is defined as follows:

$$(\text{Def. 18}) \quad \text{partdiff}(h, z, i, j) = (\text{proj}(j, n) \cdot h \cdot \text{reproj}(i, z))'((\text{proj}(i, m))(z)).$$

The following propositions are true:

- (20) Let m, n be non empty elements of $\mathbb{N}$, F be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, G be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, and y be an element of $\mathcal{R}^m$. Suppose $F = G$ and $x = y$. Then F is differentiable in x if and only if G is differentiable in y .
- (21) Let m, n be non empty elements of $\mathbb{N}$, F be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, G be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, and y be an element of $\mathcal{R}^m$. If $F = G$ and $x = y$ and F is differentiable in x , then $F'(x) = G'(y)$.
- (22) If $f = h$ and $x = z$, then $\text{Proj}(j, n) \cdot f \cdot \text{reproj}(i, x) = \langle \text{proj}(j, n) \cdot h \cdot \text{reproj}(i, z) \rangle$.
- (23) Suppose $f = h$ and $x = z$. Then f is partially differentiable in x w.r.t. i and j if and only if h is partially differentiable in z w.r.t. i and j .
- (24) If $f = h$ and $x = z$ and f is partially differentiable in x w.r.t. i and j , then $(\text{partdiff}(f, x, i, j))(\langle 1 \rangle) = \langle \text{partdiff}(h, z, i, j) \rangle$.

Let m, n be non empty elements of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and let X be a set. We say that f is partially differentiable on X w.r.t. i if and only if:

$$(\text{Def. 19}) \quad X \subseteq \text{dom } f \text{ and for every point } x \text{ of } \langle \mathcal{E}^m, \|\cdot\| \rangle \text{ such that } x \in X \text{ holds } f|_X \text{ is partially differentiable in } x \text{ w.r.t. } i.$$

We now state the proposition

- (25) If f is partially differentiable on X w.r.t. i , then X is a subset of $\langle \mathcal{E}^m, \|\cdot\| \rangle$.

Let m, n be non empty elements of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and let us consider X . Let us assume that f is partially differentiable on X w.r.t. i . The functor $f|_X$ yielding a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to the real norm space of bounded linear operators from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$ is defined by:

(Def. 20) $\text{dom}(f|X) = X$ and for every point x of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $x \in X$ holds $(f|X)_x = \text{partdiff}(f, x, i)$.

The following propositions are true:

- (26) $(f_1 + f_2) \cdot \text{reproj}(i, x) = f_1 \cdot \text{reproj}(i, x) + f_2 \cdot \text{reproj}(i, x)$ and $(f_1 - f_2) \cdot \text{reproj}(i, x) = f_1 \cdot \text{reproj}(i, x) - f_2 \cdot \text{reproj}(i, x)$.
- (27) $r(f \cdot \text{reproj}(i, x)) = (r f) \cdot \text{reproj}(i, x)$.
- (28) Suppose f_1 is partially differentiable in x w.r.t. i and f_2 is partially differentiable in x w.r.t. i . Then $f_1 + f_2$ is partially differentiable in x w.r.t. i and $\text{partdiff}(f_1 + f_2, x, i) = \text{partdiff}(f_1, x, i) + \text{partdiff}(f_2, x, i)$.
- (29) Suppose g_1 is partially differentiable in y w.r.t. i and g_2 is partially differentiable in y w.r.t. i . Then $g_1 + g_2$ is partially differentiable in y w.r.t. i and $\text{partdiff}(g_1 + g_2, y, i) = \text{partdiff}(g_1, y, i) + \text{partdiff}(g_2, y, i)$.
- (30) Suppose f_1 is partially differentiable in x w.r.t. i and f_2 is partially differentiable in x w.r.t. i . Then $f_1 - f_2$ is partially differentiable in x w.r.t. i and $\text{partdiff}(f_1 - f_2, x, i) = \text{partdiff}(f_1, x, i) - \text{partdiff}(f_2, x, i)$.
- (31) Suppose g_1 is partially differentiable in y w.r.t. i and g_2 is partially differentiable in y w.r.t. i . Then $g_1 - g_2$ is partially differentiable in y w.r.t. i and $\text{partdiff}(g_1 - g_2, y, i) = \text{partdiff}(g_1, y, i) - \text{partdiff}(g_2, y, i)$.
- (32) Suppose f is partially differentiable in x w.r.t. i . Then $r f$ is partially differentiable in x w.r.t. i and $\text{partdiff}(r f, x, i) = r \cdot \text{partdiff}(f, x, i)$.
- (33) Suppose g is partially differentiable in y w.r.t. i . Then $r g$ is partially differentiable in y w.r.t. i and $\text{partdiff}(r g, y, i) = r \cdot \text{partdiff}(g, y, i)$.

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