

Riemann Indefinite Integral of Functions of Real Variable¹

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Summary. In this article we define the Riemann indefinite integral of functions of real variable and prove the linearity of that [1]. And we give some examples of the indefinite integral of some elementary functions. Furthermore, also the theorem about integral operation and uniform convergent sequence of functions is proved.

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The papers [24], [25], [3], [23], [5], [13], [2], [26], [7], [21], [8], [10], [4], [17], [16], [15], [14], [19], [20], [6], [9], [11], [18], [12], [27], and [22] provide the terminology and notation for this paper.

1. PRELIMINARIES

For simplicity, we adopt the following rules: a, b, r are real numbers, A is a non empty set, X, x are sets, f, g, F, G are partial functions from $\mathbb{R}$ to $\mathbb{R}$, and n is an element of $\mathbb{N}$.

Next we state a number of propositions:

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- (1) Let f, g be functions from A into $\mathbb{R}$. Suppose $\text{rng } f$ is upper bounded and $\text{rng } g$ is upper bounded and for every set x such that $x \in A$ holds $|f(x) - g(x)| \leq a$. Then $\sup \text{rng } f - \sup \text{rng } g \leq a$ and $\sup \text{rng } g - \sup \text{rng } f \leq a$.
- (2) Let f, g be functions from A into $\mathbb{R}$. Suppose $\text{rng } f$ is lower bounded and $\text{rng } g$ is lower bounded and for every set x such that $x \in A$ holds $|f(x) - g(x)| \leq a$. Then $\inf \text{rng } f - \inf \text{rng } g \leq a$ and $\inf \text{rng } g - \inf \text{rng } f \leq a$.
- (3) If $f|X$ is bounded on X , then f is bounded on X .
- (4) For every real number x such that $x \in X$ and $f|X$ is differentiable in x holds f is differentiable in x .
- (5) If $f|X$ is differentiable on X , then f is differentiable on X .
- (6) Suppose f is differentiable on X and g is differentiable on X . Then $f + g$ is differentiable on X and $f - g$ is differentiable on X and fg is differentiable on X .
- (7) If f is differentiable on X , then rf is differentiable on X .
- (8) Suppose for every set x such that $x \in X$ holds $g(x) \neq 0$ and f is differentiable on X and g is differentiable on X . Then $\frac{f}{g}$ is differentiable on X .
- (9) If for every set x such that $x \in X$ holds $f(x) \neq 0$ and f is differentiable on X , then $\frac{1}{f}$ is differentiable on X .
- (10) Suppose $a \leq b$ and $[a, b] \subseteq X$ and F is differentiable on X and $F'|_X$ is integrable on $[a, b]$ and $F'|_X$ is bounded on $[a, b]$. Then $F(b) = \int_a^b (F'|_X)(x)dx + F(a)$.

2. THE DEFINITION OF INDEFINITE INTEGRAL

Let X be a set and let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. The functor $\text{IntegralFuncs}(f, X)$ yields a set and is defined by the condition (Def. 1).

- (Def. 1) $x \in \text{IntegralFuncs}(f, X)$ if and only if there exists a partial function F from $\mathbb{R}$ to $\mathbb{R}$ such that $x = F$ and F is differentiable on X and $F'|_X = f|X$.

Let X be a set and let F, f be partial functions from $\mathbb{R}$ to $\mathbb{R}$. We say that F is an integral of f on X if and only if:

- (Def. 2) $F \in \text{IntegralFuncs}(f, X)$.

The following propositions are true:

- (11) If F is an integral of f on X , then $X \subseteq \text{dom } F$.
- (12) Suppose F is an integral of f on X and G is an integral of g on X . Then $F + G$ is an integral of $f + g$ on X and $F - G$ is an integral of $f - g$ on X .

- (13) If F is an integral of f on X , then rF is an integral of rf on X .
- (14) If F is an integral of f on X and G is an integral of g on X , then FG is an integral of $fG + Fg$ on X .
- (15) Suppose for every set x such that $x \in X$ holds $G(x) \neq 0$ and F is an integral of f on X and G is an integral of g on X . Then $\frac{F}{G}$ is an integral of $\frac{fG - Fg}{G^2}$ on X .
- (16) Suppose that
- (i) $a \leq b$,
 - (ii) $[a, b] \subseteq \text{dom } f$,
 - (iii) f is continuous on $[a, b]$,
 - (iv) $]a, b[\subseteq \text{dom } F$, and
- (v) for every real number x such that $x \in]a, b[$ holds $F(x) = \int_a^x f(x)dx + F(a)$.
- Then F is an integral of f on $]a, b[$.
- (17) Let x, x_0 be real numbers. Suppose f is continuous on $[a, b]$ and $x \in]a, b[$ and $x_0 \in]a, b[$ and F is an integral of f on $]a, b[$. Then $F(x) = \int_{x_0}^x f(x)dx + F(x_0)$.
- (18) Suppose $a \leq b$ and $[a, b] \subseteq X$ and F is an integral of f on X and f is integrable on $[a, b]$ and f is bounded on $[a, b]$. Then $F(b) = \int_a^b f(x)dx + F(a)$.
- (19) Suppose $a \leq b$ and $[a, b] \subseteq X$ and f is continuous on X . Then f is continuous on $[a, b]$ and f is integrable on $[a, b]$ and f is bounded on $[a, b]$.
- (20) If $a \leq b$ and $[a, b] \subseteq X$ and f is continuous on X and F is an integral of f on X , then $F(b) = \int_a^b f(x)dx + F(a)$.
- (21) Suppose that $b \leq a$ and $[b, a] \subseteq X$ and f is integrable on $[b, a]$ and g is integrable on $[b, a]$ and f is bounded on $[b, a]$ and g is bounded on $[b, a]$ and $X \subseteq \text{dom } f$ and $X \subseteq \text{dom } g$ and F is an integral of f on X and G is an integral of g on X . Then $F(a) \cdot G(a) - F(b) \cdot G(b) = \int_b^a (fG)(x)dx + \int_b^a (Fg)(x)dx$.
- (22) Suppose that $b \leq a$ and $[b, a] \subseteq X$ and $X \subseteq \text{dom } f$ and $X \subseteq \text{dom } g$ and

f is continuous on X and g is continuous on X and F is an integral of f on X and G is an integral of g on X . Then $F(a) \cdot G(a) - F(b) \cdot G(b) = \int_b^a (f G)(x)dx + \int_b^a (F g)(x)dx$.

3. EXAMPLES OF INDEFINITE INTEGRAL

We now state several propositions:

(23) The function $\sin$ is an integral of the function $\cos$ on $\mathbb{R}$.

(24) $(\text{The function } \sin)(b) - (\text{the function } \sin)(a) = \int_a^b (\text{the function } \cos)(x)dx$.

(25) $(-1)(\text{the function } \cos)$ is an integral of the function $\sin$ on $\mathbb{R}$.

(26) $(\text{The function } \cos)(a) - (\text{the function } \cos)(b) = \int_a^b (\text{the function } \sin)(x)dx$.

(27) The function $\exp$ is an integral of the function $\exp$ on $\mathbb{R}$.

(28) $(\text{The function } \exp)(b) - (\text{the function } \exp)(a) = \int_a^b (\text{the function } \exp)(x)dx$.

(29) $\frac{n+1}{\mathbb{Z}}$ is an integral of $(n+1)\frac{n}{\mathbb{Z}}$ on $\mathbb{R}$.

(30) $(\frac{n+1}{\mathbb{Z}})(b) - (\frac{n+1}{\mathbb{Z}})(a) = \int_a^b ((n+1)\frac{n}{\mathbb{Z}})(x)dx$.

4. UNIFORM CONVERGENT FUNCTIONAL SEQUENCE

We now state the proposition

(31) Let H be a sequence of partial functions from $\mathbb{R}$ into $\mathbb{R}$ and r_1 be a sequence of real numbers. Suppose that

(i) $a < b$,

(ii) for every element n of $\mathbb{N}$ holds $H(n)$ is integrable on $[a, b]$ and $H(n)$

is bounded on $[a, b]$ and $r_1(n) = \int_a^b H(n)(x)dx$, and

(iii) H is uniform-convergent on $[a, b]$.

Then $\lim_{[a, b]} H$ is bounded on $[a, b]$ and $\lim_{[a, b]} H$ is integrable on $[a, b]$

and r_1 is convergent and $\lim r_1 = \int_a^b \lim_{[a, b]} H(x)dx$.

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