

Some Properties of Some Special Matrices. Part II

Xiaopeng Yue
 Qingdao University of Science
 and Technology
 China

Dahai Hu
 Qingdao University of Science
 and Technology
 China

Xiquan Liang
 Qingdao University of Science
 and Technology
 China

Summary. This article provides definitions of idempotent, nilpotent, involutory, self-reversible, similar, and congruent matrices, the trace of a matrix and their main properties.

MML identifier: MATRIX_8, version: 7.6.01 4.53.937

The terminology and notation used here are introduced in the following articles: [7], [3], [1], [9], [8], [6], [4], [2], [5], [11], and [10].

We adopt the following convention: n is a natural number, K is a field, and $M_1, M_2, M_3, M_4, M_5, M_6$ are matrices over K of dimension n .

Let n be a natural number, let K be a field, and let M_1 be a matrix over K of dimension n . We say that M_1 is idempotent if and only if:

(Def. 1) $M_1 \cdot M_1 = M_1$.

We say that M_1 is 2-nilpotent if and only if:

(Def. 2) $M_1 \cdot M_1 = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{n \times n}^K$.

We say that M_1 is involutory if and only if:

$$(Def. 3) \quad M_1 \cdot M_1 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{K}^{n \times n}.$$

We say that M_1 is self invertible if and only if:

$$(Def. 4) \quad M_1 \text{ is invertible and } M_1^\sim = M_1.$$

We now state a number of propositions:

- (1) $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{K}^{n \times n}$ is idempotent and involutory.
- (2) If $n > 0$, then $\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{K}^{n \times n}$ is idempotent and 2-nilpotent.
- (3) If $n > 0$ and $M_2 = M_1^T$, then M_1 is idempotent iff M_2 is idempotent.
- (4) If M_1 is involutory, then M_1 is invertible.
- (5) If M_1 is idempotent and M_2 is idempotent and M_1 is permutable with M_2 , then $M_1 \cdot M_1$ is permutable with $M_2 \cdot M_2$.
- (6) If $n > 0$ and M_1 is idempotent and M_2 is idempotent and M_1 is permutable with M_2 and $M_1 \cdot M_2 = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{K}^{n \times n}$, then $M_1 + M_2$ is idempotent.
- (7) If $n > 0$ and M_1 is idempotent and M_2 is idempotent and $M_1 \cdot M_2 = -M_2 \cdot M_1$, then $M_1 + M_2$ is idempotent.
- (8) If M_1 is idempotent and M_2 is invertible, then $M_2^\sim \cdot M_1 \cdot M_2$ is idempotent.
- (9) If $n > 0$ and M_1 is invertible and idempotent, then $M_1^\sim$ is idempotent.
- (10) If M_1 is invertible and idempotent, then $M_1 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{K}^{n \times n}$.
- (11) If M_1 is idempotent and M_2 is idempotent and M_1 is permutable with M_2 , then $M_1 \cdot M_2$ is idempotent.
- (12) If $n > 0$ and M_1 is idempotent and M_2 is idempotent and M_1 is permutable with M_2 and $M_3 = M_1^T \cdot M_2^T$, then M_3 is idempotent.
- (13) If M_1 is idempotent and M_2 is idempotent and M_1 is invertible, then $M_1 \cdot M_2$ is idempotent.
- (14) If $n > 0$ and M_1 is idempotent and orthogonal, then M_1 is symmetrical.

(15) If M_1 is idempotent and M_2 is idempotent and $M_2 \cdot M_1 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{n \times n}^K$, then $M_1 \cdot M_2$ is idempotent.

(16) If M_1 is idempotent and orthogonal, then $M_1 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{n \times n}^K$.

(17) If $n > 0$ and M_1 is symmetrical and $M_2 = M_1^T$, then $M_1 \cdot M_2$ is symmetrical.

(18) If $n > 0$ and M_1 is symmetrical and $M_2 = M_1^T$, then $M_2 \cdot M_1$ is symmetrical.

(19) If M_1 is invertible and $M_1 \cdot M_2 = M_1 \cdot M_3$, then $M_2 = M_3$.

(20) If M_1 is invertible and $M_2 \cdot M_1 = M_3 \cdot M_1$, then $M_2 = M_3$.

(21) If $n > 0$ and M_1 is invertible and $M_2 \cdot M_1 = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{n \times n}^K$, then

$$M_2 = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{n \times n}^K.$$

(22) If $n > 0$ and M_1 is invertible and $M_2 \cdot M_1 = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{n \times n}^K$, then

$$M_2 = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{n \times n}^K.$$

(23) If M_1 is 2-nilpotent and permutable with M_2 and $n > 0$, then $M_1 \cdot M_2$ is 2-nilpotent.

(24) If $n > 0$ and M_1 is 2-nilpotent and M_2 is 2-nilpotent and M_1 is permutable with M_2 and $M_1 \cdot M_2 = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{n \times n}^K$, then $M_1 + M_2$ is 2-nilpotent.

(25) If M_1 is 2-nilpotent and M_2 is 2-nilpotent and $M_1 \cdot M_2 = -M_2 \cdot M_1$ and $n > 0$, then $M_1 + M_2$ is 2-nilpotent.

(26) If M_1 is 2-nilpotent and $M_2 = M_1^T$ and $n > 0$, then M_2 is 2-nilpotent.

$$(27) \quad \text{If } M_1 \text{ is 2-nilpotent and idempotent, then } M_1 = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{K}^{n \times n}.$$

$$(28) \quad \text{If } n > 0, \text{ then } \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{K}^{n \times n} \neq \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{K}^{n \times n}.$$

$$(29) \quad \text{If } n > 0 \text{ and } M_1 \text{ is 2-nilpotent, then } M_1 \text{ is not invertible.}$$

$$(30) \quad \text{If } M_1 \text{ is self invertible, then } M_1 \text{ is involutory.}$$

$$(31) \quad \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{K}^{n \times n} \text{ is self invertible.}$$

$$(32) \quad \text{If } M_1 \text{ is self invertible and idempotent, then } M_1 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{K}^{n \times n}.$$

$$(33) \quad \text{If } M_1 \text{ is self invertible and symmetrical, then } M_1 \text{ is orthogonal.}$$

Let n be a natural number, let K be a field, and let M_1, M_2 be matrices over K of dimension n . We say that M_1 is similar to M_2 if and only if:

(Def. 5) There exists a matrix M over K of dimension n such that M is invertible and $M_1 = M^\sim \cdot M_2 \cdot M$.

Let us notice that the predicate M_1 is similar to M_2 is reflexive and symmetric.

The following propositions are true:

$$(34) \quad \text{If } M_1 \text{ is similar to } M_2 \text{ and } M_2 \text{ is similar to } M_3 \text{ and } n > 0, \text{ then } M_1 \text{ is similar to } M_3.$$

$$(35) \quad \text{If } M_1 \text{ is similar to } M_2 \text{ and } M_2 \text{ is idempotent, then } M_1 \text{ is idempotent.}$$

$$(36) \quad \text{If } M_1 \text{ is similar to } M_2 \text{ and } M_2 \text{ is 2-nilpotent and } n > 0, \text{ then } M_1 \text{ is 2-nilpotent.}$$

$$(37) \quad \text{If } M_1 \text{ is similar to } M_2 \text{ and } M_2 \text{ is involutory, then } M_1 \text{ is involutory.}$$

$$(38) \quad \text{If } M_1 \text{ is similar to } M_2 \text{ and } n > 0, \text{ then } M_1 + \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{K}^{n \times n} \text{ is similar to } M_2 + \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{K}^{n \times n}.$$

$$(39) \quad \text{If } M_1 \text{ is similar to } M_2 \text{ and } n > 0, \text{ then } M_1 + M_1 \text{ is similar to } M_2 + M_2.$$

$$(40) \quad \text{If } M_1 \text{ is similar to } M_2 \text{ and } n > 0, \text{ then } M_1 + M_1 + M_1 \text{ is similar to } M_2 + M_2 + M_2.$$

- (41) If M_1 is invertible, then $M_2 \cdot M_1$ is similar to $M_1 \cdot M_2$.
- (42) If M_2 is invertible and M_1 is similar to M_2 and $n > 0$, then M_1 is invertible.
- (43) If M_2 is invertible and M_1 is similar to M_2 and $n > 0$, then $M_1^\sim$ is similar to $M_2^\sim$.

Let n be a natural number, let K be a field, and let M_1, M_2 be matrices over K of dimension n . We say that M_1 is congruent to M_2 if and only if:

- (Def. 6) There exists a matrix M over K of dimension n such that M is invertible and $M_1 = M^T \cdot M_2 \cdot M$.

Next we state several propositions:

- (44) If $n > 0$, then M_1 is congruent to M_1 .
- (45) If M_1 is congruent to M_2 and $n > 0$, then M_2 is congruent to M_1 .
- (46) If M_1 is congruent to M_2 and M_2 is congruent to M_3 and $n > 0$, then M_1 is congruent to M_3 .
- (47) If M_1 is congruent to M_2 and $n > 0$, then $M_1 + M_1$ is congruent to $M_2 + M_2$.
- (48) If M_1 is congruent to M_2 and $n > 0$, then $M_1 + M_1 + M_1$ is congruent to $M_2 + M_2 + M_2$.
- (49) If M_1 is orthogonal, then $M_2 \cdot M_1$ is congruent to $M_1 \cdot M_2$.
- (50) If M_2 is invertible and M_1 is congruent to M_2 and $n > 0$, then M_1 is invertible.
- (51) If M_2 is invertible and M_1 is congruent to M_2 and $n > 0$ and $M_5 = M_1^T$ and $M_6 = M_2^T$, then M_5 is congruent to M_6 .
- (52) If M_4 is orthogonal and $M_1 = M_4^T \cdot M_2 \cdot M_4$, then M_1 is similar to M_2 .

Let n be a natural number, let K be a field, and let M be a matrix over K of dimension n . The functor $\text{Trace}(M)$ yields an element of K and is defined by:

- (Def. 7) $\text{Trace}(M) = \sum (\text{the diagonal of } M)$.

The following propositions are true:

- (53) If $M_2 = M_1^T$, then $\text{Trace}(M_1) = \text{Trace}(M_2)$.
- (54) $\text{Trace}(M_1 + M_2) = \text{Trace}(M_1) + \text{Trace}(M_2)$.
- (55) $\text{Trace}(M_1 + M_2 + M_3) = \text{Trace}(M_1) + \text{Trace}(M_2) + \text{Trace}(M_3)$.
- (56) $\text{Trace}\left(\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{n \times n}^K\right) = 0_K$.
- (57) If $n > 0$, then $\text{Trace}(-M_1) = -\text{Trace}(M_1)$.
- (58) If $n > 0$, then $-\text{Trace}(-M_1) = \text{Trace}(M_1)$.
- (59) If $n > 0$, then $\text{Trace}(M_1 + -M_1) = 0_K$.

- (60) If $n > 0$, then $\text{Trace}(M_1 - M_2) = \text{Trace}(M_1) - \text{Trace}(M_2)$.
- (61) If $n > 0$, then $\text{Trace}((M_1 - M_2) + M_3) = (\text{Trace}(M_1) - \text{Trace}(M_2)) + \text{Trace}(M_3)$.
- (62) If $n > 0$, then $\text{Trace}((M_1 + M_2) - M_3) = (\text{Trace}(M_1) + \text{Trace}(M_2)) - \text{Trace}(M_3)$.
- (63) If $n > 0$, then $\text{Trace}(M_1 - M_2 - M_3) = \text{Trace}(M_1) - \text{Trace}(M_2) - \text{Trace}(M_3)$.

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Received January 4, 2006
