

Combinatorial Grassmannians

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Summary. In the paper I construct the configuration G which is a partial linear space. It consists of k -element subsets of some base set as points and $(k + 1)$ -element subsets as lines. The incidence is given by inclusion. I also introduce automorphisms of partial linear spaces and show that automorphisms of G are generated by permutations of the base set.

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The articles [15], [17], [3], [14], [7], [11], [13], [8], [18], [19], [4], [12], [16], [9], [5], [6], [10], [2], and [1] provide the notation and terminology for this paper.

1. PRELIMINARIES

We follow the rules: k, n denote elements of $\mathbb{N}$ and X, Y, Z denote sets.

One can prove the following propositions:

- (1) For all sets a, b such that $a \neq b$ and $\overline{\overline{a}} = n$ and $\overline{\overline{b}} = n$ holds $\overline{\overline{a \cap b}} < n$ and $n + 1 \leq \overline{\overline{a \cup b}}$.
- (2) For all sets a, b such that $\overline{\overline{a}} = n + k$ and $\overline{\overline{b}} = n + k$ holds $\overline{\overline{a \cap b}} = n$ iff $\overline{\overline{a \cup b}} = n + 2 \cdot k$.
- (3) $\overline{\overline{X}} \leq \overline{\overline{Y}}$ iff there exists a function f such that f is one-to-one and $X \subseteq \text{dom } f$ and $f^\circ X \subseteq Y$.
- (4) For every function f such that f is one-to-one and $X \subseteq \text{dom } f$ holds $\overline{\overline{f^\circ X}} = \overline{\overline{X}}$.
- (5) If $X \setminus Y = X \setminus Z$ and $Y \subseteq X$ and $Z \subseteq X$, then $Y = Z$.
- (6) Let Y be a non empty set and p be a function from X into Y . Suppose p is one-to-one. Let x_1, x_2 be subsets of X . If $x_1 \neq x_2$, then $p^\circ x_1 \neq p^\circ x_2$.

- (7) Let a, b, c be sets such that $\overline{a} = n - 1$ and $\overline{b} = n - 1$ and $\overline{c} = n - 1$ and $\overline{a \cap b} = n - 2$ and $\overline{a \cap c} = n - 2$ and $\overline{b \cap c} = n - 2$ and $2 \leq n$. Then
- (i) if $3 \leq n$, then $\overline{a \cap b \cap c} = n - 2$ and $\overline{a \cup b \cup c} = n + 1$ or $\overline{a \cap b \cap c} = n - 3$ and $\overline{a \cup b \cup c} = n$, and
 - (ii) if $n = 2$, then $\overline{a \cap b \cap c} = n - 2$ and $\overline{a \cup b \cup c} = n + 1$.
- (8) Let P_1, P_2 be projective incidence structures. Suppose the projective incidence structure of P_1 = the projective incidence structure of P_2 . Let A_1 be a point of P_1 and A_2 be a point of P_2 . Suppose $A_1 = A_2$. Let L_1 be a line of P_1 and L_2 be a line of P_2 . If $L_1 = L_2$, then if A_1 lies on L_1 , then A_2 lies on L_2 .
- (9) Let P_1, P_2 be projective incidence structures. Suppose the projective incidence structure of P_1 = the projective incidence structure of P_2 . Let A_1 be a subset of the points of P_1 and A_2 be a subset of the points of P_2 . Suppose $A_1 = A_2$. Let L_1 be a line of P_1 and L_2 be a line of P_2 . If $L_1 = L_2$, then if A_1 lies on L_1 , then A_2 lies on L_2 .

Let us note that there exists a projective incidence structure which is linear, up-2-rank, and strict and has non-trivial-lines.

2. CONFIGURATION G

A partial linear space is an up-2-rank projective incidence structure with non-trivial-lines.

Let k be an element of $\mathbb{N}$ and let X be a non empty set. Let us assume that $0 < k$ and $k + 1 \leq \overline{X}$. The functor $G_k(X)$ yields a strict partial linear space and is defined by the conditions (Def. 1).

- (Def. 1)(i) The points of $G_k(X) = \{A; A \text{ ranges over subsets of } X: \overline{A} = k\}$,
- (ii) the lines of $G_k(X) = \{L; L \text{ ranges over subsets of } X: \overline{L} = k + 1\}$, and
 - (iii) the incidence of $G_k(X) = \subseteq_{2^X} \cap \{ \text{the points of } G_k(X), \text{ the lines of } G_k(X) \}$.

One can prove the following four propositions:

- (10) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $0 < k$ and $k + 1 \leq \overline{X}$. Let A be a point of $G_k(X)$ and L be a line of $G_k(X)$. Then A lies on L if and only if $A \subseteq L$.
- (11) For every element k of $\mathbb{N}$ and for every non empty set X such that $0 < k$ and $k + 1 \leq \overline{X}$ holds $G_k(X)$ is Vebleian.
- (12) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $0 < k$ and $k + 1 \leq \overline{X}$. Let $A_1, A_2, A_3, A_4, A_5, A_6$ be points of $G_k(X)$ and L_1, L_2, L_3, L_4 be lines of $G_k(X)$. Suppose that A_1 lies on L_1 and A_2 lies on L_1 and A_3 lies on L_2 and A_4 lies on L_2 and A_5 lies on L_1 and A_5 lies on

L_2 and A_1 lies on L_3 and A_3 lies on L_3 and A_2 lies on L_4 and A_4 lies on L_4 and A_5 does not lie on L_3 and A_5 does not lie on L_4 and $L_1 \neq L_2$ and $L_3 \neq L_4$. Then there exists a point A_6 of $G_k(X)$ such that A_6 lies on L_3 and A_6 lies on L_4 and $A_6 = A_1 \cap A_2 \cup A_3 \cap A_4$.

- (13) For every element k of $\mathbb{N}$ and for every non empty set X such that $0 < k$ and $k + 1 \leq \overline{\overline{X}}$ holds $G_k(X)$ is Desarguesian.

Let S be a projective incidence structure and let K be a subset of the points of S . We say that K is a clique if and only if:

- (Def. 2) For all points A, B of S such that $A \in K$ and $B \in K$ there exists a line L of S such that $\{A, B\}$ lies on L .

Let S be a projective incidence structure and let K be a subset of the points of S . We say that K is a maximal-clique if and only if:

- (Def. 3) K is a clique and for every subset U of the points of S such that U is a clique and $K \subseteq U$ holds $U = K$.

Let k be an element of $\mathbb{N}$, let X be a non empty set, and let T be a subset of the points of $G_k(X)$. We say that T is a star if and only if:

- (Def. 4) There exists a subset S of X such that $\overline{\overline{S}} = k - 1$ and $T = \{A; A \text{ ranges over subsets of } X: \overline{\overline{A}} = k \wedge S \subseteq A\}$.

We say that T is a top if and only if:

- (Def. 5) There exists a subset S of X such that $\overline{\overline{S}} = k + 1$ and $T = \{A; A \text{ ranges over subsets of } X: \overline{\overline{A}} = k \wedge A \subseteq S\}$.

Next we state two propositions:

- (14) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $2 \leq k$ and $k + 2 \leq \overline{\overline{X}}$. Let K be a subset of the points of $G_k(X)$. If K is a star or a top, then K is a maximal-clique.
- (15) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $2 \leq k$ and $k + 2 \leq \overline{\overline{X}}$. Let K be a subset of the points of $G_k(X)$. If K is a maximal-clique, then K is a star or a top.

3. AUTOMORPHISMS

Let S_1, S_2 be projective incidence structures. We consider maps between projective spaces S_1 and S_2 as systems

$\langle \text{a point-map, a line-map} \rangle$,

where the point-map is a function from the points of S_1 into the points of S_2 and the line-map is a function from the lines of S_1 into the lines of S_2 .

Let S_1, S_2 be projective incidence structures, let F be a map between projective spaces S_1 and S_2 , and let a be a point of S_1 . The functor $F(a)$ yields a point of S_2 and is defined as follows:

(Def. 6) $F(a) = (\text{the point-map of } F)(a)$.

Let S_1, S_2 be projective incidence structures, let F be a map between projective spaces S_1 and S_2 , and let L be a line of S_1 . The functor $F(L)$ yields a line of S_2 and is defined by:

(Def. 7) $F(L) = (\text{the line-map of } F)(L)$.

Next we state the proposition

- (16) Let S_1, S_2 be projective incidence structures and F_1, F_2 be maps between projective spaces S_1 and S_2 . Suppose for every point A of S_1 holds $F_1(A) = F_2(A)$ and for every line L of S_1 holds $F_1(L) = F_2(L)$. Then the map of $F_1 =$ the map of F_2 .

Let S_1, S_2 be projective incidence structures and let F be a map between projective spaces S_1 and S_2 . We say that F preserves incidence strongly if and only if:

(Def. 8) For every point A_1 of S_1 and for every line L_1 of S_1 holds A_1 lies on L_1 iff $F(A_1)$ lies on $F(L_1)$.

The following proposition is true

- (17) Let S_1, S_2 be projective incidence structures and F_1, F_2 be maps between projective spaces S_1 and S_2 . Suppose the map of $F_1 =$ the map of F_2 . If F_1 preserves incidence strongly, then F_2 preserves incidence strongly.

Let S be a projective incidence structure and let F be a map between projective spaces S and S . We say that F is automorphism if and only if:

(Def. 9) The line-map of F is bijective and the point-map of F is bijective and F preserves incidence strongly.

Let S_1, S_2 be projective incidence structures, let F be a map between projective spaces S_1 and S_2 , and let K be a subset of the points of S_1 . The functor $F^\circ K$ yielding a subset of the points of S_2 is defined by:

(Def. 10) $F^\circ K = (\text{the point-map of } F)^\circ K$.

Let S_1, S_2 be projective incidence structures, let F be a map between projective spaces S_1 and S_2 , and let K be a subset of the points of S_2 . The functor $F^{-1}(K)$ yielding a subset of the points of S_1 is defined as follows:

(Def. 11) $F^{-1}(K) = (\text{the point-map of } F)^{-1}(K)$.

Let X be a set and let A be a finite set. The functor $\uparrow(A, X)$ yielding a subset of 2^X is defined as follows:

(Def. 12) $\uparrow(A, X) = \{B; B \text{ ranges over subsets of } X: \overline{\overline{B}} = \text{card } A + 1 \wedge A \subseteq B\}$.

Let k be an element of $\mathbb{N}$ and let X be a non empty set. Let us assume that $0 < k$ and $k + 1 \leq \overline{\overline{X}}$. Let A be a finite set. Let us assume that $\overline{\overline{A}} = k - 1$ and $A \subseteq X$. The functor $\uparrow(A, X, k)$ yields a subset of the points of $G_k(X)$ and is defined as follows:

(Def. 13) $\uparrow(A, X, k) = \uparrow(A, X)$.

The following propositions are true:

- (18) Let S_1, S_2 be projective incidence structures, F be a map between projective spaces S_1 and S_2 , and K be a subset of the points of S_1 . Then $F^\circ K = \{B; B \text{ ranges over points of } S_2: \bigvee_{A: \text{point of } S_1} (A \in K \wedge F(A) = B)\}$.
- (19) Let S_1, S_2 be projective incidence structures, F be a map between projective spaces S_1 and S_2 , and K be a subset of the points of S_2 . Then $F^{-1}(K) = \{A; A \text{ ranges over points of } S_1: \bigvee_{B: \text{point of } S_2} (B \in K \wedge F(A) = B)\}$.
- (20) Let S be a projective incidence structure, F be a map between projective spaces S and S , and K be a subset of the points of S . If F preserves incidence strongly and K is a clique, then $F^\circ K$ is a clique.
- (21) Let S be a projective incidence structure, F be a map between projective spaces S and S , and K be a subset of the points of S . Suppose F preserves incidence strongly and the line-map of F is onto and K is a clique. Then $F^{-1}(K)$ is a clique.
- (22) Let S be a projective incidence structure, F be a map between projective spaces S and S , and K be a subset of the points of S . Suppose F is automorphism and K is a maximal-clique. Then $F^\circ K$ is a maximal-clique and $F^{-1}(K)$ is a maximal-clique.
- (23) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $2 \leq k$ and $k+2 \leq \overline{X}$. Let F be a map between projective spaces $G_k(X)$ and $G_k(X)$. Suppose F is automorphism. Let K be a subset of the points of $G_k(X)$. If K is a star, then $F^\circ K$ is a star and $F^{-1}(K)$ is a star.

Let k be an element of $\mathbb{N}$ and let X be a non empty set. Let us assume that $0 < k$ and $k+1 \leq \overline{X}$. Let s be a permutation of X . The functor $\text{incprojmap}(k, s)$ yielding a strict map between projective spaces $G_k(X)$ and $G_k(X)$ is defined as follows:

- (Def. 14) For every point A of $G_k(X)$ holds $(\text{incprojmap}(k, s))(A) = s^\circ A$ and for every line L of $G_k(X)$ holds $(\text{incprojmap}(k, s))(L) = s^\circ L$.

One can prove the following propositions:

- (24) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $k = 1$ and $k+1 \leq \overline{X}$. Let F be a map between projective spaces $G_k(X)$ and $G_k(X)$. Suppose F is automorphism. Then there exists a permutation s of X such that the map of $F = \text{incprojmap}(k, s)$.
- (25) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $1 < k$ and $\overline{X} = k+1$. Let F be a map between projective spaces $G_k(X)$ and $G_k(X)$. Suppose F is automorphism. Then there exists a permutation s of X such that the map of $F = \text{incprojmap}(k, s)$.
- (26) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $0 < k$

and $k + 1 \leq \overline{\overline{X}}$. Let T be a subset of the points of $G_k(X)$ and S be a subset of X . If $\overline{\overline{S}} = k - 1$ and $T = \{A; A \text{ ranges over subsets of } X: \overline{\overline{A}} = k \wedge S \subseteq A\}$, then $S = \bigcap T$.

(27) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $0 < k$ and $k + 1 \leq \overline{\overline{X}}$. Let T be a subset of the points of $G_k(X)$. Suppose T is a star. Let S be a subset of X . If $S = \bigcap T$, then $\overline{\overline{S}} = k - 1$ and $T = \{A; A \text{ ranges over subsets of } X: \overline{\overline{A}} = k \wedge S \subseteq A\}$.

(28) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $0 < k$ and $k + 1 \leq \overline{\overline{X}}$. Let T_1, T_2 be subsets of the points of $G_k(X)$. If T_1 is a star and T_2 is a star and $\bigcap T_1 = \bigcap T_2$, then $T_1 = T_2$.

(29) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $0 < k$ and $k + 1 \leq \overline{\overline{X}}$. Let A be a finite subset of X . If $\overline{\overline{A}} = k - 1$, then $\uparrow(A, X, k)$ is a star.

(30) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $0 < k$ and $k + 1 \leq \overline{\overline{X}}$. Let A be a finite subset of X . If $\overline{\overline{A}} = k - 1$, then $\bigcap \uparrow(A, X, k) = A$.

(31) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $0 < k$ and $k + 3 \leq \overline{\overline{X}}$. Let F be a map between projective spaces $G_{(k+1)}(X)$ and $G_{(k+1)}(X)$. Suppose F is automorphism. Then there exists a map H between projective spaces $G_k(X)$ and $G_k(X)$ such that

- (i) H is automorphism,
- (ii) the line-map of H = the point-map of F , and
- (iii) for every point A of $G_k(X)$ and for every finite set B such that $B = A$ holds $H(A) = \bigcap (F^\circ \uparrow(B, X, k + 1))$.

(32) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $0 < k$ and $k + 3 \leq \overline{\overline{X}}$. Let F be a map between projective spaces $G_{(k+1)}(X)$ and $G_{(k+1)}(X)$. Suppose F is automorphism. Let H be a map between projective spaces $G_k(X)$ and $G_k(X)$. Suppose that

- (i) H is automorphism,
- (ii) the line-map of H = the point-map of F , and
- (iii) for every point A of $G_k(X)$ and for every finite set B such that $B = A$ holds $H(A) = \bigcap (F^\circ \uparrow(B, X, k + 1))$.

Let f be a permutation of X . If the map of $H = \text{incprojmap}(k, f)$, then the map of $F = \text{incprojmap}(k + 1, f)$.

(33) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $2 \leq k$ and $k + 2 \leq \overline{\overline{X}}$. Let F be a map between projective spaces $G_k(X)$ and $G_k(X)$. Suppose F is automorphism. Then there exists a permutation s of X such that the map of $F = \text{incprojmap}(k, s)$.

(34) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $0 < k$

and $k + 1 \leq \overline{\overline{X}}$. Let s be a permutation of X . Then $\text{incprojmap}(k, s)$ is automorphism.

- (35) Let X be a non empty set. Suppose $0 < k$ and $k+1 \leq \overline{\overline{X}}$. Let F be a map between projective spaces $G_k(X)$ and $G_k(X)$. Then F is automorphism if and only if there exists a permutation s of X such that the map of $F = \text{incprojmap}(k, s)$.

REFERENCES

- [1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(2):377–382, 1990.
- [2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(1):41–46, 1990.
- [3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(1):91–96, 1990.
- [4] Grzegorz Bancerek. Zermelo theorem and axiom of choice. *Formalized Mathematics*, 1(2):265–267, 1990.
- [5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [7] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(1):47–53, 1990.
- [8] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(1):165–167, 1990.
- [9] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(1):35–40, 1990.
- [10] Wojciech Leńczuk and Krzysztof Prażmowski. Incidence projective spaces. *Formalized Mathematics*, 2(2):225–232, 1991.
- [11] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(1):147–152, 1990.
- [12] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics*.
- [13] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(1):115–122, 1990.
- [14] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(1):25–34, 1990.
- [15] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(1):9–11, 1990.
- [16] Wojciech A. Trybulec. Axioms of incidence. *Formalized Mathematics*, 1(1):205–213, 1990.
- [17] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [18] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.
- [19] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(1):181–186, 1990.

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